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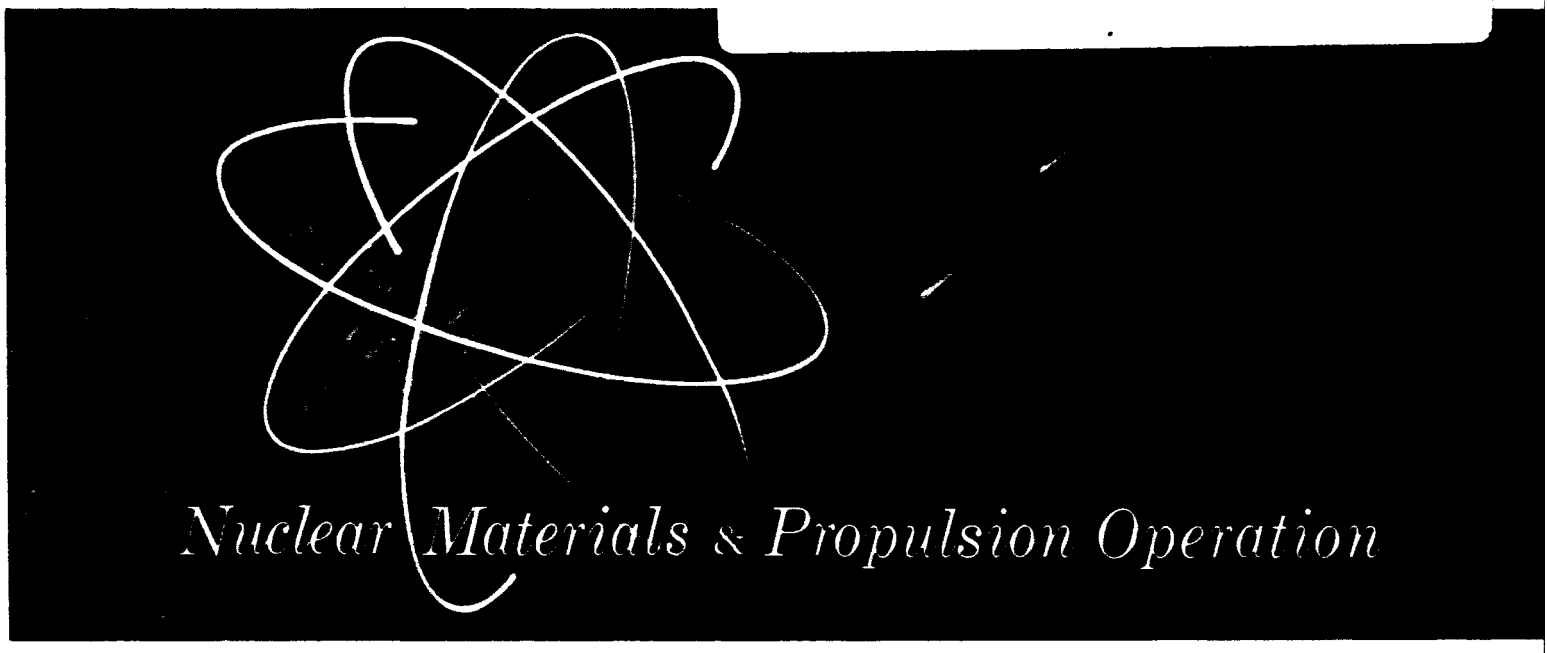

SUMMARY OF DIFFUSION THEORY FOR HOMOGENEOUS AND HETEROGENEOUS NUCLEAR REACTORS

Lectures 3 and 4

J. R. Beeler, Jr.

FLIGHT PROPULSION LABORATORY DEPARTMENT

GENERAL ELECTRIC

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J. R. Beeler, Jr. [1963] rejd

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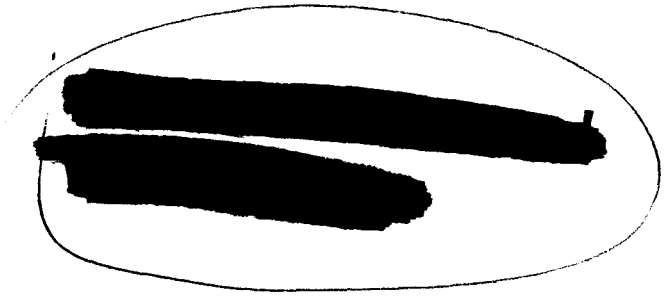

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NOMENCLATURE

Symbol	Description	Introduced*
C	Criticality factor	3
ℓ	Generation time	3
t	Time, sec	
ν	Average neutron production per fission	4
f(E)	Prompt neutron energy distribution	4
F(E)	Integral of f(E)	4
$<E>$	Average energy	4
A_w	Atomic weight	4
N_a, N	Atoms (nuclei) per cm^3	4
ρ	Mass density (pure material)	4
σ	Microscopic cross section	5.1
σ_s	Microscopic scattering cross section	5.1
σ_a	Microscopic absorption cross section	5.1
Σ	Macroscopic cross section	5.2
λ	Neutron mean free path	5.3
p(x)	Free path distribution	5.3
$<x>$	Average free path	5.3
Σ_s	Macroscopic scattering cross section	5.4
Σ_a	Macroscopic absorption cross section	5.4
Φ	Neutron flux, cm/cm^3-sec	6.0, 8.8
Σ_f	Macroscopic fission cross section	6.0
σ_f	Microscopic fission cross section	6.0
Σ_{nfa}	Macroscopic non-fission absorption cross section	6.0
σ_{nfa}	Microscopic non-fission absorption cross section	6.0

*Section Number

NOMENCLATURE (Cont.)

Symbol	Description	Introduced*
eV	Electron volt	7.0
v	Velocity, cm/sec	7.0
m_n	Neutron mass	7.0
T	Temperature, ^oK	7.0
J	Neutron current vector, neutrons/cm^2-sec	7.1
D	Neutron diffusion coefficient, cm	7.1
grad, ∇	Gradient operator	7.1
grad,^{2} ∇^2	Laplacian operator	7.1
n	Neutrons per cm^3	7.1
S	Neutron source, neutrons/cm^3-sec	7.1
S	Fuel element surface area	11.0 and 13.0
η	Fission neutrons produced per thermal neutron absorbed in fuel material	7.1
f	Thermal utilization	7.1
k, k_∞	Multiplication factor	7.1
B_m^2	Material buckling	7.1
B_g^2	Geometrical buckling	8.1
L	Diffusion length	7.1
p	Resonance escape probability	7.2
ϵ	Fast fission factor	7.2
P	Non-leakage probability	7.2
P_t	Thermal neutron non-leakage probability	7.2
P_f	Fast neutron non-leakage probability	7.2
$\Phi(\vec{r}, E)$	Neutron flux in space and energy	8.1, 8.8

*Section Number

NOMENCLATURE (Cont.)

Symbol	Description	Introduced*
$\psi(\vec{r})$	Space part of $\Phi(\vec{r}, E)$	8.1
ν'	(See Section 8.2)	8.2
H	Extrapolated slab thickness	8.3
h, h_o	Slab thickness	8.3, 10.2
d	Extrapolation distance	8.1
d	Disadvantage factor	11.1, 11.3
x, y, z	Distance, position	8.3
R	Extrapolated sphere, cylinder radius	8.5, 8.6
$R, r, \vec{r}$	Distance, position	
J_o	Bessel function, first kind	8.6
κ	Inverse diffusion length	8.8
n		
n(E)		
$n(\vec{r}, E)$	Neutron densities	8.8
$n(\vec{r}, E, \vec{\Omega})$		
Φ		
$\Phi(\vec{r})$	Neutron scalar flux	8.8
$\Phi(E)$		
$\Phi(\vec{r}, E)$		
$\vec{\Omega}$	Direction vector	8.8
K(r)	Displacement Kernel	9.1
τ	Age, slowing down area	9.2
α	(See Section 9.2)	9.2
A	Nuclear mass number	9.2
ξ	Average lethargy gain per collision	9.2

*Section Number

NOMENCLATURE (Cont.)

Symbol	Description	Introduced*
u	Lethargy	9.2
M^2	Migration area	9.3
δ	Reflector savings	10.1
Φ_c	Core flux	10.2
Φ_r	Reflector flux	10.2
Φ_1	Fast flux	10.3
Φ_2	Thermal flux	10.3
μ_n^2 ν^2	(See Section 10.3)	10.3
V_o	Fuel volume	11.1
V_1	Moderator volume	11.1
$<\Phi_o>$	Average flux in fuel element	11.1
$<\Phi_1>$	Average flux in moderator	11.1
$K_o(Kr)$	Bessel functions	11.1
$K_1(Kr)$		
X	Excess absorption	11.1
R_a	Relative absorption	11.1
f_R	Resonance utilization factor	11.2
$(\sigma_{ao})_{eff}$	(See Section 12)	12.0
I_{eff}	Effective resonance integral	12.0
VAF	Volume advantage factor	12.0
M	Fuel mass	13.0
ψ^*	Importance function	Appendix 2
W	Statistical weight	Appendix 2
δC	Variation in C	Appendix 2
$\delta\Sigma_a$	Variation in Σ_a	Appendix 2

*Section Number

NOMENCLATURE (Cont.)

Symbol	Description	Introduced*
δD	Variation in D	Appendix 2
$\delta\Sigma_f$	Variation in Σ_f	Appendix 2

*Section Number

1. INTRODUCTION

This part of the lecture series consists of two lectures on the theory of homogeneous and heterogeneous nuclear reactors. The diffusion theory model will be the main topic of discussion. I will talk about the ideas on which elementary reactor theory is based and show a list of results for the one-group and two-group diffusion theory approximations. You will be able to do a rudimentary design of a slow reactor on the basis of these lectures and the content of the lecture notes. Judging from the experience of my students at the University of Cincinnati, it will take you from 20 to 60 hours to first complete an acceptable, elementary design analysis. (See Section 15.0)

No proofs or derivations will be given in this coverage but the lecture notes point out where proofs and derivations can be found. In a practical sense it is appropriate to leave out detailed considerations both in the lectures and the notes. Reactor theory, like any other technical topic, can only be learned through intensive self-study and the working of lots of problems. This set of two, short lectures cannot possibly give you a detailed understanding of reactor theory. It is therefore meant to serve two particular purposes: (1) It should teach you what are the important concepts in reactor theory and indicate classical references you can read if you feel an urge to understand these concepts. (2) It should provide you with a list of steps to follow in the execution of a rudimentary reactor analysis if you either feel the urge or are constrained to compute without the advantage of understanding what you are doing.

These notes are based upon a relatively small number of sources (four). These sources are listed below together with an alphabetic designation symbol which is used in the text to indicate the source of a particular statement or to point out where one can obtain a more comprehensive coverage of a particular topic. For example, (D, 129-134;156-159) means "look at pages 129-134 and 156-159 in Neutron Transport Theory by B. Davison".

Symbol	Reference
D	B. Davison, Neutron Transport Theory (Book), Oxford Press (1957)
WW	A. M. Weinberg and E. P. Wigner, Physical Theory of Neutron Chain Reactors (Book), University of Chicago Press (1959) Second Impression

Symbol	Reference
GE	S. Glasstone and M. C. Edlund, Elements of Nuclear Reactor Theory (Book), Van Nostrand Company (1958) Seventh Printing
CPH	Case, Placzek and de Hoffman, Theory of Neutron Diffusion Vol. I (1952), U. S. Chamber of Commerce

Each of these references enjoys an excellent reputation and can be found in any good technical library. CPH contains extensive tabular results and graphs of important functions in diffusion theory. After reading GE, one should go on to D, WW, and CPH to refine his understanding of reactors and neutron transport. D and CPH require a knowledge of complex variables.

2. CHAIN REACTOR (WW, 168-180; 1-18)

The first concept to learn is the idea of a chain reactor. A chain reactor involves two essential parts: (1) fuel and (2) chain carriers. In a chain reactor, the interaction of a chain carrier with the fuel must produce the following: (1) the liberation of energy and (2) new chain carriers. Chain carriers are produced anew by the same energy-liberating process which they induce when they interact with the fuel.

A chain reaction is self-sustaining if and only if the rate at which chain carriers are produced is greater or equal to the rate at which chain carriers are expended. If the production and expenditure rates are equal, the reactor is said to be critical and in this state its power output is constant.

Our concern will be with nuclear chain reactors. In this specific instance, the fuel is a collection of fissionable nuclei and the chain carriers are neutrons. The energy-liberating reaction is fissioning of the fuel nuclei which is induced by neutron absorption. Hence, each fission event in a nuclear reaction corresponds to the expenditure of one neutron, i.e., one chain carrier.

3. CRITICALITY (WW, 168-180)

A number called the criticality factor is used to represent the degree to which a nuclear chain reactor is self-sustaining. Many people take it for granted that the criticality factor, C, is a constant but this is not true - the criticality factor need not be constant in the general sense. In general, the criticality factor is defined as the ratio of the neutron production rate to the neutron loss rate. This defines the so called dynamic criticality factor. When the reactor is critical, C is a constant by definition, i. e., $C = 1$ for a critical reactor. But when the reactor is not in the critical state, the neutron production and loss rates contain energy dependent transients and C is time dependent.

Generation Model - For the present we will use the idea of criticality given by the generation model. This will work as long as we deal only with critical systems or near-critical systems, i. e., $0.9 < C < 1.1$. In the generation model, one assumes that the reactor process can be described in terms of a number of successive neutron generations wherein the n-th generation of neutrons is solely responsible for the production of the $(n + 1)$-th generation. This is a typical biological picture. In this model, the criticality factor is tacitly assumed to be a constant and is defined to be the ratio of the neutron populations in two successive generations.

Specifically, if N_n represents the total neutron population for the n-th generation then the criticality factor is

$$C = N_n/N_{n-1} \tag{3.1}$$

and

$$N_n = N_1\, C^{n-1} \tag{3.2}$$

where N_1 is the neutron population of the first generation. Now this way of looking at things assumes that it is meaningful to ascribe a temporal separation between successive generations which is also a constant. This time interval is denoted by ℓ and is called the generation time. In this context, the time interval between the first and n-th generation is $(n-1)\ell$. And, if one considers the first generation as existing at time zero, the time of the n-th generation is $t_n = (n - 1)\ell$. This identification allows one to get a neutron production rate equation which involves only the criticality factor and the neutron population at a time t. The result is

$$\begin{aligned} dN/dt &= N(\log_e C)/\ell \\ &\cong N(C - 1)/\ell \quad \text{(if C is close to unity)} \end{aligned} \tag{3.3}$$

From this equation one finds that the neutron population at time t is

$$N(t) = N(t = 0) \exp [(C - 1)t/\ell] \qquad (3.4)$$

The generation time for thermal reactors is about 0.1 sec. Hence, if $C = 1.005$, for example, the reactor power level will increase by $\exp(0.05) = 1.06$ in one second and 1.65 in 10 seconds. This is a sufficient time for automatic control systems to dampen a power excursion and return the reactor to a critical state. If $C = 0.995$ the reactor power level will decrease by 1.06 in one second.

When the criticality factor is less than unity, the reactor is said to be subcritical; when the criticality factor is greater than unity the reactor is said to be supercritical. The power output of a non-critical reactor is a function of time; only when the reactor is critical is its power output constant. In order to increase the power level of a reactor from P_1 to $P_2 > P_1$, one allows it to go supercritical, for a time sufficient for the fission rate to increase to the desired level P_2, and then returns it to the critical state. It will then provide a constant power output P_2. The power level is reduced by allowing the reactor to go subcritical until the power level falls to the value desired. (See above numerical example for $C = 1.005$ and $C = 0.995$.)

4. NEUTRON REACTIONS WITH NUCLEI (D, 1-14)

We will discuss the reactions of neutrons with nuclei in a very abbreviated manner. Only those aspects of direct concern to reactor calculations will be mentioned. For our purposes there exist two general types of neutron-nucleus reactions. One of these is the absorption reaction and the other is the scattering reaction. A neutron is absorbed by a nucleus and disappears from the scene in an absorption reaction. Excepting the instance wherein neutron absorption induces fissioning of fuel nuclei, we will not concern ourselves with the possible consequences of neutron absorption other than that it represents the loss of one chain carrier. In the scattering reaction, a neutron collides with a nucleus, transfers kinetic energy to this nucleus and then bounces off, somewhat less energetic, and in a different direction from that in which it was traveling prior to the scattering collision. If the neutron scatters elastically, both momentum and kinetic energy are conserved during the collision. If the neutron scatters inelastically, momentum is conserved but kinetic energy is not conserved, and a gamma ray is emitted from the target nucleus. All neutron scattering is assumed to be elastic scattering in these notes. We will not discuss inelastic scattering.

One particular type of absorption reaction, namely, the fission reaction, is clearly important in reactor theory; without the fission process nuclear reaction could not exist. The fission process consists of a nucleus splitting into fragments. Any nucleus can be made to fission if it is struck by a sufficiently energetic neutron. The fissioning of uranium is the principal fission reaction used in nuclear reactors. Uranium nuclei are the reactor fuel. Uranium-235 (U^{235}) undergoes fission upon absorption of both fast and slow neutrons, and, in fact, is more likely to fission the smaller the neutron energy. U^{235} requires fast neutrons with energies of at least 1 meV to induce fission to an appreciable extent. In the fission of a U^{235} nucleus induced by slow neutrons, several neutrons are emitted. Table 4.1a lists the probabilities for the emission of n neutrons per fission in U^{235}. The average number of fission neutrons is denoted by the symbol ν. $\nu \cong 2.5$ for U^{235}. Tables 4.1b and 4.1c list ν, microscopic cross sections (see Section 5) and η (see Section 7) for U, U^{235}, Pu^{239} and U^{233}.

About 0.9925 of all fission neutrons emitted are ejected within a time interval of 10^{-14} sec. These fission neutrons are called prompt neutrons. Prompt neutrons are emitted from the highly excited fission fragments into which the uranium splits upon fissioning. The remaining fraction (0.0075) of the fission neutrons are emitted by the daughters of the radioactive fission fragments. These neutrons appear anywhere from fractions of a second to minutes after the fission event and are called delayed neutrons. Table 4.2 lists the six delayed neutron groups for fissioning of

TABLE 4. 1(a)

PROBABILITIES FOR EMISSION OF n NEUTRONS PER FISSION IN U^{235} (WW, 114)

Neutron Energy	n						
	0	1	2	3	4	5	$\nu = n_{ave}$
80 keV	0. 02	0. 17	0. 36	0. 31	0. 12	0. 03	2. 45
1. 25 meV	0. 02	0. 11	0. 30	0. 41	0. 10	0. 06	2. 65

TABLE 4. 1(b)

THERMAL* CROSS SECTIONS, AND η AND ν FOR U^{235} NATURAL U AND Pu^{239} (WW, 124)

	U^{235}	U	Pu^{239}
σ_s	10 barns	8. 3 barns	9. 7 barns
σ_a	697 barns	7. 7 barns	1025 barns
σ_i	579 barns	4. 2 barns	738 barns
η	2. 07	1. 34	2. 09
ν	2. 47		2. 91

*0. 025 eV neutrons

TABLE 4. 1(c)

η AND ν FOR 1 meV NEUTRONS (WW, 129)

	U^{235}	U^{233}	Pu^{239}
η	2. 3	2. 45	2. 7
ν	2. 65	2. 7	3. 0

U^{235} and their total yield. Although the delayed neutron yield is small, their long delay times serve to increase the average "generation" time, ℓ, from the value 10^{-3} sec, which would occur if only prompt neutrons were emitted, to about 10^{-1} sec. The existence of delayed neutrons, therefore, makes reactor control possible.

Problem: If only prompt neutrons were emitted, by what factor would the reactor power level increase during a one-second excursion with $C = 1.005$?

Answer: $e^5 \cong 150$.

The energy distribution of prompt neutrons is

$$f(E) = 0.484 \exp(-E) \sinh\sqrt{2E} \quad \text{(See Figure 4.1)} \qquad (4.1)$$

The meaning of f(E) is as follows: The fraction of all prompt neutrons emitted with energies in the range dE at E is f(E)dE. The average prompt neutron energy $<E>$ is

$$<E> = \int_0^\infty Ef(E)dE = 2.0 \text{ meV} \qquad (4.2)$$

The most probable energy is 0.72 meV and one-half of all prompt neutrons emitted have energies below 1.6 meV. The fraction F(E) of all prompt neutrons emitted which have energies less than E is plotted in Figure 4.2. Nearly all fission neutrons are emitted with energies below 10 meV. When the fission spectrum is mentioned in textbooks and technical articles it is to the prompt neutron spectrum that the authors refer.

The power output of a reactor is determined by the number of fission reactions occurring per second. The energy release per fission event is about 200 meV which corresponds to 3.2×10^{-11} watt-sec. Hence 3.1×10^{10} fission events per second are required to produce 1 watt of power.

$$3.1 \times 10^{10} \text{ fissions/sec} = 1 \text{ watt of power} \qquad (4.3)$$

On this basis, the energy supplied when all U^{235} nuclei contained in one gram of U^{235} undergo fission is approximately one megawatt-day.

Remarks:

1. $1 \text{ eV} = 1.6 \times 10^{-12}$ ergs $= 1.6 \times 10^{-19}$ watt-sec.
2. $1 \text{ meV} = 1.6 \times 10^{-6}$ ergs $= 1.6 \times 10^{-13}$ watt-sec.

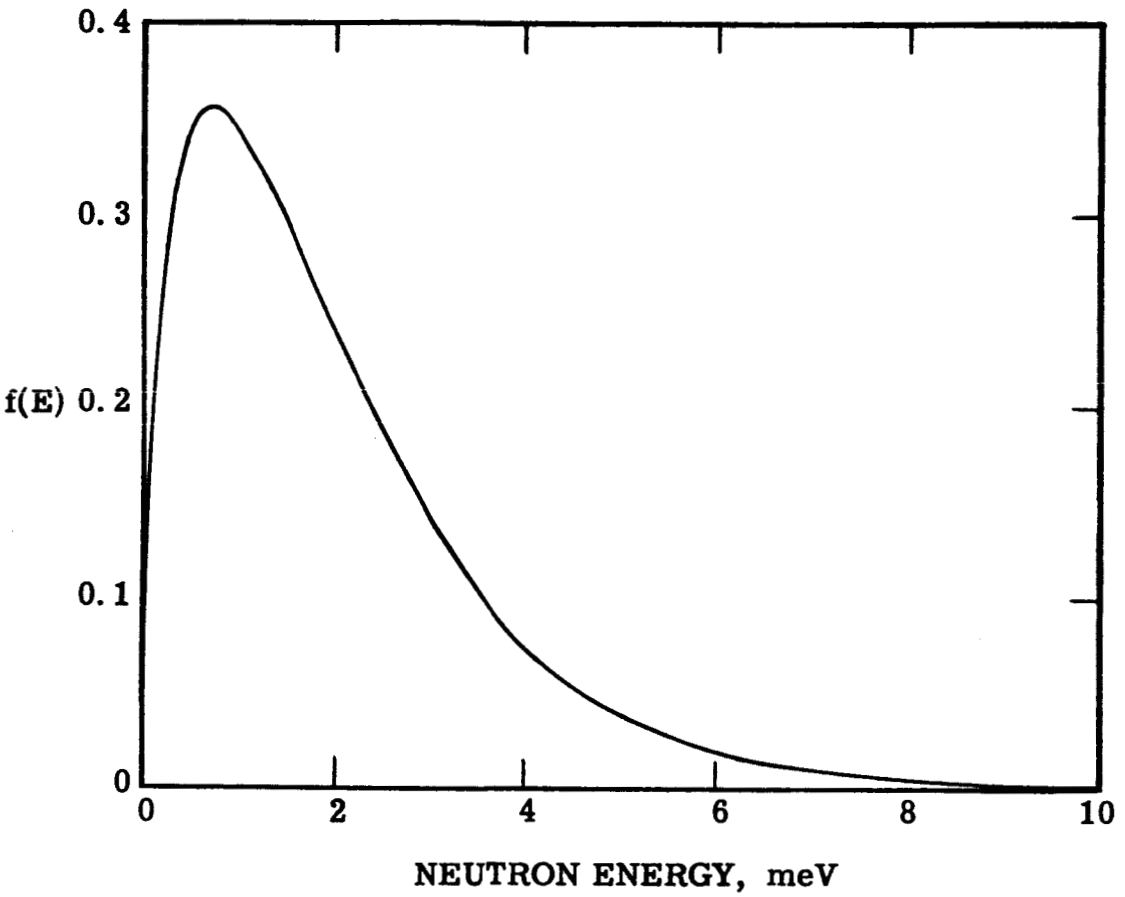


Fig. 4.1 - Prompt neutron energy distribution f(E). This distribution (spectrum) applies to U^{235} and Pu^{239}

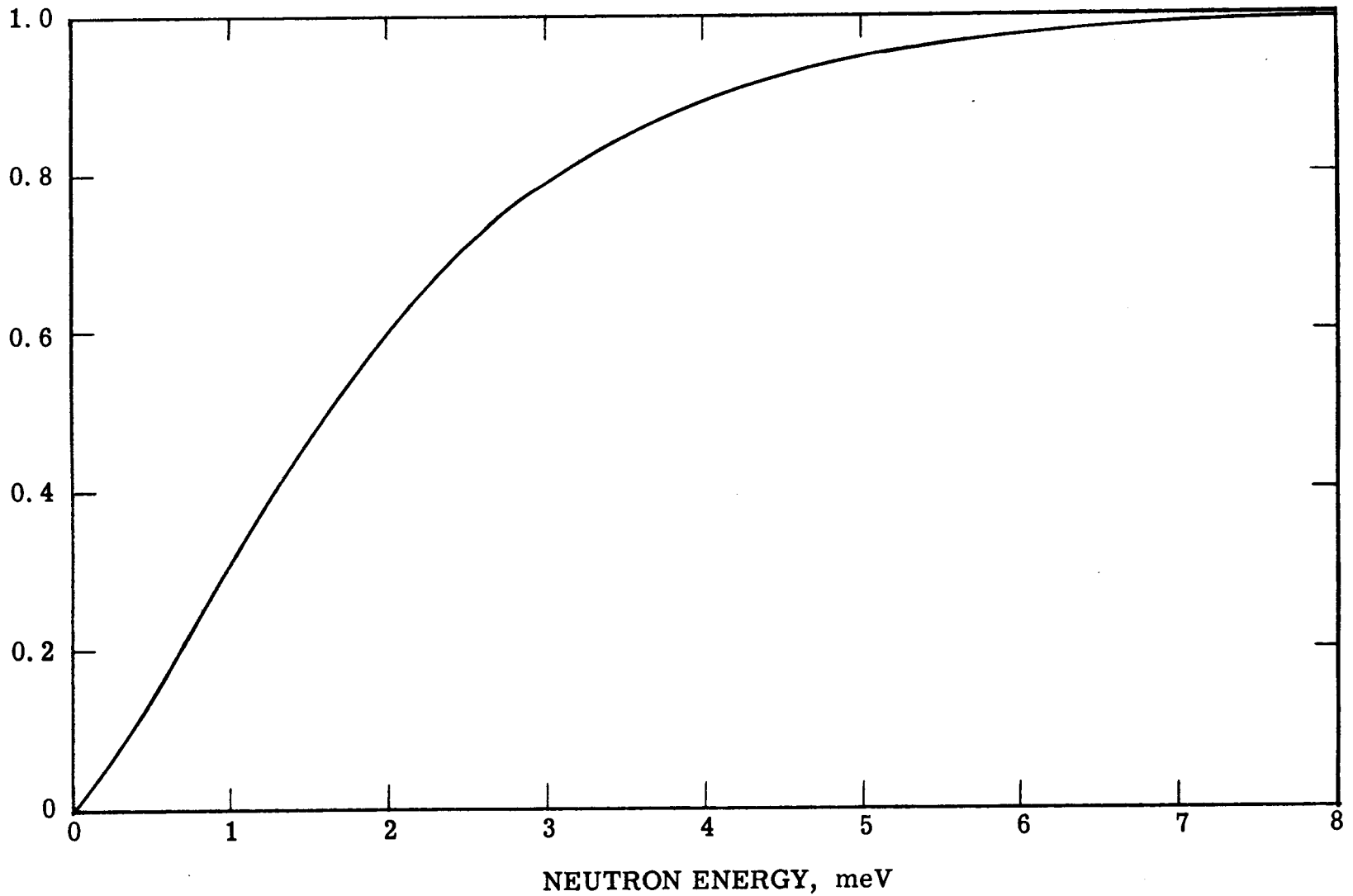


Fig. 4.2 - F(E) is the fraction of all prompt neutrons emitted with energies less than E. F(E) is the integral of f(E') over the interval $0 < E' < E$.

3. The number of atoms N_a of a given element with atomic weight A_w contained in one cm^3 of material is

$$N_a = \frac{m}{A_w} \times 6.025 \times 10^{23} \text{ per } cm^3 \qquad (4.4)$$

where m is the mass of the given element per cm^3 of material. In a monoatomic material $m = \rho$, the mass density.

TABLE 4.2

DELAYED NEUTRON CHARACTERISTICS (WW, 136)

Half-Life (Sec)	Decay Constant (Sec^{-1})	Energy (Mev)	Number Per 100 Fission Neutrons				
			Th^{232}	U^{233}	U^{235}	U^{238}	Pu^{239}
54	0.0128	0.25	0.085	0.020	0.03	0.015	0.01
22	0.0315	.56	0.35	.075	.18	.17	.06
5.6	0.125	.43	0.45	.105	.22	.28	.045
2.12	0.325	.62	1.20	.075	.23	.71	.085
0.45	1.55	0.42	0.45	0.025	.07	.42	0.03
0.15	4.5	-	0.09	-	0.02	0.15	-
			2.6	0.30	0.75	1.75	0.23
Number of delayed neutrons per 100 fissions			6.3	0.78	1.80	4.4	0.67

TABLE 4.3

CONSTITUTION OF NATURAL URANIUM (WW, 5)

	U^{238}	U^{235}	U^{234}
Abundance*	99.28%	0.71%	0.0058%
Half-life	4.51×10^9 yr	7.1×10^8 yr	2.6×10^5 yr

*Weight percent

TABLE 4.4

TYPES OF NUCLEAR REACTORS (WW, 15)

Point of View	Type
Energy of neutrons causing fission	Thermal, intermediate, fast*
Fuel	Natural U, U^{235}, U^{238}, Pu^{239}
Moderator	H_2O, D_2O, Be, C, BeO, Be_2C
Cooling System	Conduction, circulating coolant, circulating fuel, boiling
Structure	Homogeneous, heterogeneous, solid or liquid materials

*See Table 7.1 for associated neutron energies.

5. CROSS SECTIONS AND MEAN FREE PATH (D, 1-14)

When a neutron strikes a nucleus the type of reaction which will occur is not absolutely determined in the sense that one can say "this particular reaction will certainly be a scattering reaction" or "this particular reaction will certainly be an absorption reaction." However, it is possible to state the probability that a particular neutron-nucleus collision will result in a scattering reaction or in an absorption reaction. The practice is to represent the probability that a particular type of reaction will occur by stating the cross section for that reaction. Two types of cross sections are used, the microscopic cross section, σ, and the macroscopic cross section, Σ. The microscopic cross section refers to neutron collisions with individual nuclei of a specific type while the macroscopic cross section is used to specify the collision probability of neutrons in bulk material which may be made up of several different nuclear types. The microscopic cross section depends only upon the neutron energy and the type of nucleus involved. The macroscopic cross section depends, in addition, upon the number of nuclei contained in a unit volume of material.

5.1 MICROSCOPIC CROSS SECTION

The microscopic cross section for a particular type of reaction is defined in terms of the number of neutron-nucleus reactions induced in a thin sheet of target material by a monoenergetic neutron beam which strikes the target sheet at normal incidence. In this definition the target material contains only one nuclear species.

$$\sigma = \frac{\text{(number of neutron-nucleus reactions/cm}^2\text{-sec)}}{\text{(number of nuclei/cm}^2\text{) x (number of incident neutrons/cm}^2\text{-sec)}}$$

$$\sigma = \frac{\text{Fraction of target nuclei reacting per second}}{\text{Number of incident neutrons/cm}^2\text{-sec}}$$

The dimensions of the microscopic cross section are cm^2, the dimensions of an area. The unit used to specify microscopic cross sections is the barn; one barn is 10^{-24} cm^2.

Let σ_s and σ_a be the scattering and absorption cross sections for neutrons of a given energy and a given type of target nucleus. Their sum

$$\sigma = \sigma_s + \sigma_a \qquad (5.2)$$

is called the total microscopic cross section.

5.2 MACROSCOPIC CROSS SECTION

The macroscopic cross section for a particular type of reaction is defined as the product of the microscopic cross section for that reaction and the number density of nuclei, N, i.e., the number of nuclei per unit volume.

$$\Sigma = \sigma N \tag{5.3}$$

The dimensions of the macroscopic cross section are cm^{-1}. It is the probability per unit length of neutron travel for a neutron-nucleus reaction.

$$\Sigma = \text{Probability per unit length of neutron travel for a neutron-nucleus reaction.} \tag{5.4}$$

5.3 MEAN FREE PATH

The reciprocal of the macroscopic cross section is the neutron mean free path, λ.

$$\lambda = \frac{1}{\Sigma} \tag{5.5}$$

λ is the average distance between successive neutron collisions. The free path between successive neutron collisions is exponentially distributed. The normalized differential free path distribution is,

$$p(x) = \Sigma e^{-\Sigma x} = \frac{1}{\lambda}\, e^{-\frac{x}{\lambda}} \tag{5.6}$$

and

$$\int_0^\infty p(x)dx = 1$$

The meaning of $p(x)$ is as follows: the fraction of all freepaths ending in the interval dx about x is $p(x)dx$. The fraction $F(x<x_0)$ of all free paths of length x less than x_0 is,

$$F(x<x_0) = \int_0^{x_0} \Sigma e^{-\Sigma x}\, dx = 1 - e^{-\Sigma x_0} \tag{5.7}$$

$F(x<x_0)$ is the cumulative free path distribution. It is interesting to note that more than half of the free paths are less than λ as,

$$F(x<\lambda) = \int_0^{\lambda} \Sigma e^{-\Sigma x}\, dx = 1 - e^{-\Sigma\lambda} = 0.632 \qquad (5.8)$$

(Remember $\Sigma\lambda = 1$).

Problem: Compute the fraction of all free paths less than 2λ, 3λ, 4λ, 5λ, 5.3λ, 9λ. Ans. 0.845, 0.950, 0.982, 0.993, 0.995, 0.9999.

Problem: Compute the average free path over the free path range $\lambda < x < \infty$. Hint: See Remark (4) at the end of this section.

Ans. $$\langle x \rangle_{\lambda<x} = \int_{\lambda}^{\infty} xp(x)dx \Big/ \int_{\lambda}^{\infty} p(x)dx$$

$$= (2\lambda/e)\,\chi(1/e) = 2\lambda$$

Problem: Compute the average free path over the free path range $0 < x < \lambda$.

Ans. $\lambda(e-2)/(e-1) \cong 0.418\lambda$

5.4 CROSS SECTIONS FOR POLYATOMIC MATERIAL

Consider now a material composed of n different nuclear types. Let Σ_i and N_i be the total macroscopic cross section and the number density, respectively, for the i-th nuclear type. The total cross section for the material is,

$$\Sigma = \Sigma_1 N_1 + \Sigma_2 N_2 + \ldots + \Sigma_n N_n \qquad (5.9)$$

and the neutron mean free path is,

$$\lambda = 1/\Sigma$$

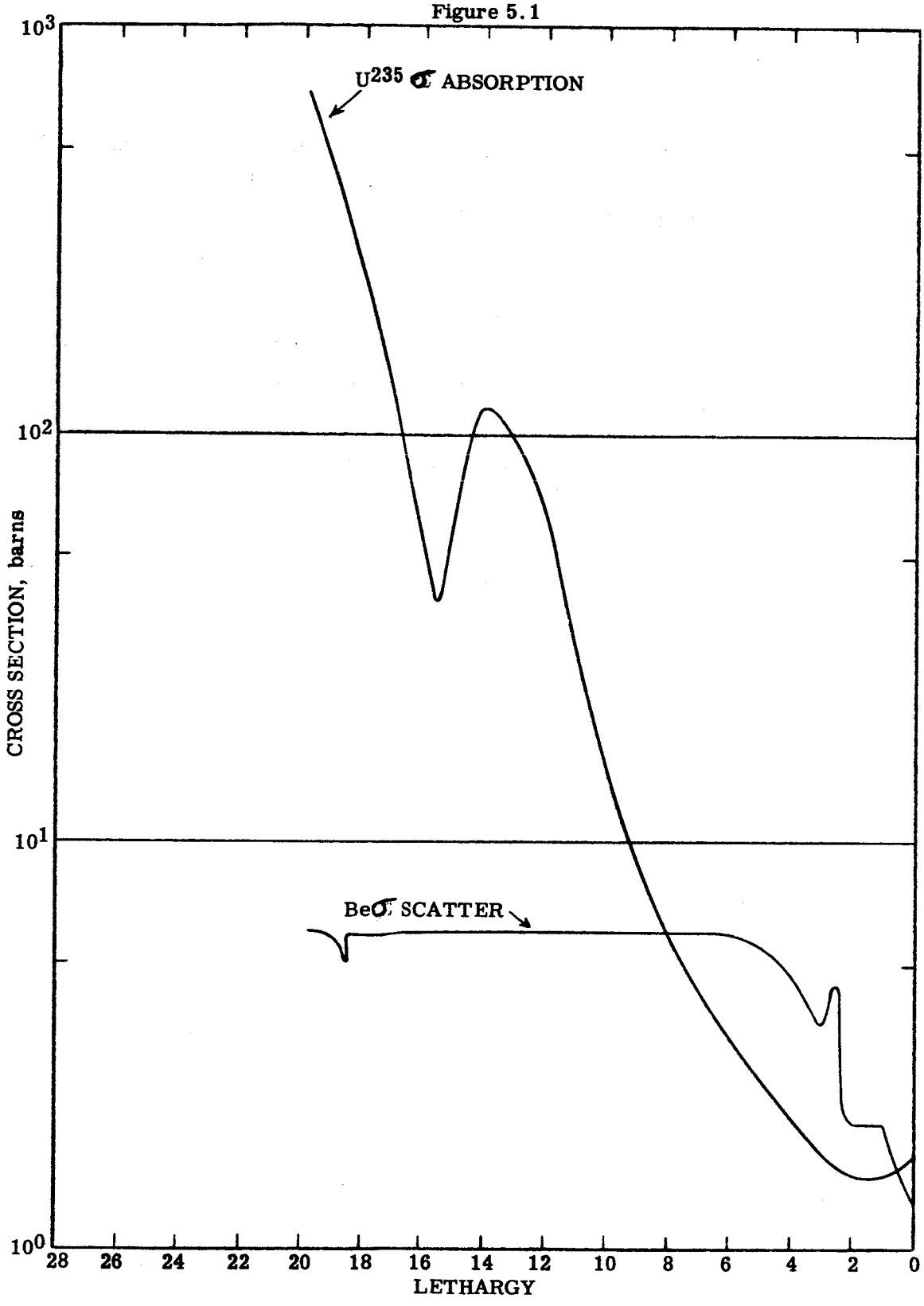
Similarly, the macroscopic scattering cross section for the material is,

$$\Sigma_s = \Sigma_{s1} N_1 + \Sigma_{s2} N_2 + \ldots + \Sigma_{sn} N_n \qquad (5.10)$$

and the macroscopic absorption cross section is,

$$\Sigma_a = \Sigma_{a1} N_1 + \Sigma_{a2} N_2 + \ldots \Sigma_{an} N_n \qquad (5.11)$$

Figure 5.1


10^3
10^2
10^1
10^0
U^{235} σ_f ABSORPTION
Beσ_s SCATTER
CROSS SECTION, barns
28 26 24 22 20 18 16 14 12 10 8 6 4 2 0
LETHARGY

The scattering mean free path is $1/\Sigma_s$ and the absorption mean free path is $1/\Sigma_a$. One sees that λ_s, λ_a, and λ are related as,

$$\lambda = \frac{\lambda_a \lambda_s}{\lambda_a + \lambda_s} \tag{5.12}$$

Remark (1): Note that cross sections are additive, like resistances in a series circuit; and that mean free paths combine like resistances in a parallel circuit.

Remark (2): λ is the average distance between successive neutron collisions.

Remark (3): λ_s is the average distance between successive neutron scattering collisions. It is always true that $\lambda_s \geq \lambda$ because $\Sigma_s \leq \Sigma$.

Remark (4): If p(x) is the normalized differential distribution for a random variable x then

$$\langle x \rangle_{a,b} = \int_a^b x\, p(x)dx \Big/ \int_a^b p(x)dx \tag{5.13}$$

6. NUMBER OF REACTIONS PER UNIT VOLUME PER SECOND (D, 41-42); (GE, 148)

Up to now we have looked at the way to describe the probability that a single neutron will experience a reaction of a particular type in a single collision event. In reactor calculations, however, one is concerned with the combined effect of all the collisions of many different neutrons with nuclei. It is possible to do reactor calculations by using the neutron number density, n, to represent the neutron population. However, in most work on reactor calculations, a quantity called the neutron flux, Φ, is used rather than the number density. The relations between n and Φ are described in Section 8. 8. For the present it is sufficient to state that the neutron flux Φ is the neutron track length traced out per unit volume in one second. This is an important concept and bears repeating... "the neutron flux is the neutron track length traced out per unit volume in one second." Is it the track length traced out by just one neutron? The answer to this question is: "No." It is the track length traced out by all neutrons in that unit volume per second. This being the case, if one multiplies the neutron flux by the macroscopic cross section the result is the number of neutron-nucleus reactions per unit volume per second, i. e., the reaction rate per unit volume. This follows because Σ is the probability for one reaction per unit length of neutron travel. The following table summarizes the reaction rate densities with which we will be concerned:

$$\begin{aligned} &\Sigma\Phi = \text{total reaction rate density} \\ &\Sigma_a\Phi = \text{absorption reaction rate density} \\ &\Sigma_s\Phi = \text{scattering reaction rate density} \\ &\Sigma_f\Phi = \text{fission absorption reaction rate density} \end{aligned} \tag{6.1}$$

Σ_a, Σ_s and Σ_f are, respectively, the absorption, scattering and fission cross sections. In a homogeneous reactor

$$\begin{aligned} \Sigma_a &= \Sigma_a^{mod} + \Sigma_a^{fuel} \\ \Sigma_s &= \Sigma_s^{mod} + \Sigma_s\,\text{fuel} \end{aligned} \tag{6.2}$$

where 'mod' and 'fuel' signify moderator and fuel materials, and

$$\Sigma_a^{fuel} = \Sigma_f^{fuel} + \Sigma_{nfa}^{fuel} \tag{6.3}$$

where 'nfa' means 'non-fission absorption.' The non-fission absorption cross section for the reactor is

$$\Sigma_{nfa} = \Sigma_a^{mod} + \Sigma_{nfa}^{fuel} \qquad (6.4)$$

Problem: $\sigma_s = 10$ barns, $\sigma_a = 600$ barns and $\sigma_f = 500$ barns. Assume an atomic weight of 235 and a mass of 19 grams per cm^3. If $\Phi = 10^{15}$ cm/cm^3-sec what is the scattering reaction rate per cm^3 and the non-fission absorption rate per cm^3? Hint: N = $(19 \times 6.025 \times 10^{23}/235)$ at/cm^3, $\Sigma_s = N\sigma_s = [19 \times 6.025 \times 10^{23}/235]$ $[10 \times 10^{-24}]$ cm^{-1}, $\sigma_{nfa} = \sigma_a - \sigma_f$.

Problem: Compute the neutron mean free path from the data given above.

Hint: $\sigma = \sigma_a + \sigma_s$

7. DIFFUSION THEORY (WW, 181-218); (D, 94-102); (GE, 90-136)

Homogeneous Infinite System

Because most neutron cross sections behave, approximately, as $\sigma \propto 1/\sqrt{E}$, the slower is the neutron the larger is its reaction probability. Hence, in order to minimize the amount of fuel required to make a reactor operate at a prescribed power level, slow neutrons are usually employed as the primary agent for inducing fission. Reactors for which this is true are called thermal reactors because the kinetic energy of the neutrons which induce most of the fission events corresponds to the thermal kinetic energy of molecules at room temperature, i. e., about 0. 0254 eV (1 eV is equivalent to 11, 600^{o}K). The kinetic energy of a neutron is KE = 1/2 $m_n v^2$ ergs where m_n is the neutron rest mass in grams and v is its velocity in cm/sec. ($m_n = 1.675 \times 10^{-24}$ grams) The following neutron energy-temperature-velocity table is useful. (Table 7.1) E is the neutron energy. (As an exercise start with column 1 and compute column 3; then do the converse. Remember 1eV = 1.602×10^{-12} ergs.) T is obtained from the relation E = kT where $k = 1.38 \times 10^{-16}$ erg/^{o}K is Boltzmann's constant. T(^{o}C) = T(^{o}K)-273^{o}K.

It can be seen from Figure 4.1, however, that most fission neutrons are emitted with energies in the meV range. A means for slowing these fast neutrons to the thermal energy level is required in thermal reactors. This slowing down or moderation, as it is called, is accomplished by mixing non-fissionable material with the fuel. This material is called moderator material. Fast neutrons slow down largely via elastic collisions with moderator nuclei. A good moderator material should have an extremely small absorption cross section and be composed of light nuclei. Light nuclei are preferable to heavy nuclei because neutrons lose a larger fraction of their energy in elastic scattering collisions the lighter is the target nucleus.

Before we concern ourselves with neutron slowing down, it will be a good idea to look briefly at a hypothetical monoenergetic reactor in order to single out the complications that exist just as a consequence of reactor composition. In this reactor, fuel and moderator nuclei are taken to be homogeneously mixed (See WW, 378, line 3) and all neutrons will be assumed to be thermal neutrons, including fission neutrons.

TABLE 7. 1

NEUTRON ENERGIES AND 'TEMPERATURES'

E(eV)	T(^{o}K)	v(cm/sec)	Type
0. 001	11. 6	4.37×10^{4}	Cold
0. 025	290	2.19×10^{5}	Thermal
0. 034	400	2.6×10^{5}	Thermal
0. 052	600	3.1×10^{5}	Thermal
0. 069	800	3.6×10^{5}	Thermal
0. 086	1000	4.0×10^{5}	Thermal
1. 0	1.16×10^{4}	1.38×10^{6}	Slow (resonance)
100	1.16×10^{6}	1.38×10^{7}	Slow
10^{4}	1.16×10^{8}	1.38×10^{8}	Intermediate
10^{6}	1.16×10^{10}	1.38×10^{9}	Fast
10^{8}	1.16×10^{12}	1.28×10^{10}	Ultrafast
10^{10}	1.16×10^{14}	2.99×10^{10}	Relativistic

7.1 MONOENERGETIC NEUTRONS

The diffusion approximation for describing monoenergetic neutron transport in an infinite system is based on the assumption that Fick's diffusion equations are valid. There are two such equations. The first one connects the neutron current $\vec{J}$ with the gradient of the neutron flux.

$$\vec{J} = - D \text{ grad } \Phi \tag{7.1}$$

$$(\text{grad } \Phi = \vec{i}\, \partial\Phi/\partial x + \vec{j}\, \partial\Phi/\partial y + \vec{k}\partial\Phi/\partial y)$$

The constant D is the neutron diffusion coefficient. For the time being we will assume D is a known quantity. The second equation states that the rate at which the number of neutrons in a unit volume (n) increases as a result of neutron diffusion is

$$(\partial n/\partial t)_D = - \text{ div } \vec{J} = D \text{ grad}^2 \Phi \tag{7.2}$$

$$\text{grad}^2 \Phi = \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2$$

These equations are valid if:

1. The magnitude of grad Φ is small
2. Neutron scattering is isotropic in the laboratory coordinate system
3. D is a constant
4. The neutron cross section is constant

From Fick's equations and our understanding of how to compute the absorption reaction rate density, $\Sigma_a\Phi$, it is possible to write down the neutron population balance equation for an infinite system. This equation is,

$$\underset{\text{(a)}}{\partial n/\partial t} = \underset{\text{(a)}}{D \text{ grad}^2 \Phi} + \underset{\text{(a)}}{S} \underset{\text{(b)}}{- \Sigma_a\Phi} \tag{7.3}$$

(a) is the rate at which neutrons are added per unit volume, S being a source term, i. e., neutrons per cm^3-sec supplied by some neutron source

(b) is the rate at which neutrons are lost per unit volume.

If $\partial n/\partial t = 0$, the neutron population is stationary and the system is critical.

A natural question, at this point, is: "What is the source term S?" One answers this question by remembering that there are $\Sigma_f\Phi$ fission reactions per unit volume per second and that an average of ν neutrons are emitted per fission event. This reasoning leads to,

$$S = \nu \Sigma_f \Phi \tag{7.4}$$

and the balance equation, for a critical system becomes

$$D \, \text{grad}^2 \Phi + \nu \Sigma_f \Phi - \Sigma_a \Phi = 0 \quad (\Sigma_a = \Sigma_a^{mod} + \Sigma_a^{fuel}) \tag{7.5}$$

dividing by D it follows that

$$\text{grad}^2 \Phi + \left[\frac{(\nu \Sigma_f - \Sigma_a)}{D}\right] \Phi = 0 \tag{7.6}$$

$$\text{grad}^2 \Phi + B_m^2 \Phi = 0 \qquad B_m^2 \equiv \frac{\nu \Sigma_f - \Sigma_a}{D} \quad (*\text{See Remark})$$

Another symbol for the operator grad^2 is ∇^2 (Laplarian) which will be used in all subsequent balance equation statements. The number B_m^2 is called the material buckling of the system as it depends only on the composition of the reactor. (When we get to finite reactors it will turn out that one designs a critical finite reactor by finding a size such that the associated geometrical buckling, B_g^2, for the finite geometry is just equal to the material buckling associated with an infinite system of the same composition. (See Section 8.1.)

By definition, the criticality factor for this infinite system is

$$C = \frac{\text{Neutron Production Rate}}{\text{Neutron Loss Rate}}$$

$$= \frac{\nu \Sigma_f \Phi}{\Sigma_a \Phi} \tag{7.7}$$

$$= \frac{\nu \Sigma_f}{\Sigma_a}$$

Conventionally, the criticality factor for this system is written as,

$$C = \left[\frac{\nu \Sigma_f}{\Sigma_a^{fuel}}\right] \left[\frac{\Sigma_a^{fuel}}{\Sigma_a}\right] = \eta f \quad (*\text{See footnote}) \tag{7.8}$$

where

$$\eta = \frac{\nu \Sigma_f}{\Sigma_a^{fuel}} \tag{7.9}$$

*In most text books the quantity ηf for an infinite monoenergetic system is denoted by k and called the multiplication factor. In an infinite system C and k are equal but in finite systems neutrons can leak out and $C < k$ whenever leakage can occur. Some texts use the notation $k_\infty = \eta f$ to emphasize that the multiplication factor for an infinite system is being talked about.

$$f = \frac{\Sigma_a^{fuel}}{\Sigma_a} \qquad (7.10)$$

η is the number of fission neutrons produced per neutron absorbed in the fuel and is a property of the fuel material alone. $\eta = 2.07$ for U^{235} and 1.34 for U (U denotes natural uranium).

f is called the thermal utilization factor. It is the fraction of all thermal neutrons absorbed which are absorbed in the fuel.

Remark:

Diffusion theory formalism uses the quantity $L^2 = D/\Sigma_a$ called the thermal neutron diffusion area or diffusion length squared. Physically, L^2 is one-sixth the mean square distance from the point at which a neutron starts diffusing to the point where it is absorbed. Using this notation one sees B_m^2 can be written as

$$B_m^2 = (k - 1)/L^2 \qquad (7.11)$$

and the balance equation then becomes

$$\nabla^2 \Phi + \left(\frac{k - 1}{L^2}\right) \Phi = 0 \qquad (7.12)$$

This form of the balance equation is used frequently.

7.2 THERMAL AND FAST NEUTRONS

Retaining an infinite geometry, to avoid the complications of neutron leakage, we will now introduce fast fission neutrons rather than assume all fission neutrons are emitted at thermal energy as we did in (7.1). In effect we now have two groups of neutrons (1) fast neutrons and (2) thermal neutrons. Some of the fast neutrons will be absorbed while slowing down to thermal energy. The fraction of all fast neutrons which escape absorption during moderation (slowing down) is represented by the symbol p. A portion of the fast neutrons absorbed during moderation will cause fission, i.e., all fissions are not caused by thermal neutrons. This effect is represented by the fast fission factor ϵ. ϵ is the ratio of number of all fission events occurring to those particular fission events induced by thermal neutrons. In thermal reactors, ϵ is barely larger than unity, in fast reactors it is, of course, quite large or they wouldn't be called fast reactors.

The source term for the thermal neutron balance equation is now

$$S = p[\epsilon \nu \Sigma_f \Phi] \tag{7.13}$$

where Φ is the thermal neutron flux and Σ_f is the thermal neutron fission cross section. The balance equation is

$$D\nabla^2 \Phi + p\epsilon\nu\Sigma_f\Phi - \Sigma_a\Phi = 0$$

$$\nabla^2\Phi + \left[\frac{p\epsilon\nu\Sigma_f - \Sigma_a}{D}\right] \Phi = 0 \tag{7.14}$$

$$\nabla^2\Phi + B_m^2\Phi = 0 \qquad B_m^2 = \frac{p\epsilon\nu\Sigma_f - \Sigma_a}{D}$$

The criticality factor is by definition

$$C = \frac{\text{Thermal Neutron Production Rate}}{\text{Thermal Neutron Loss Rate}} \tag{7.15}$$

$$= \frac{p\epsilon\nu\Sigma_f\Phi}{\Sigma_a\Phi} = p\epsilon\eta f$$

(remember that $\eta = \nu\Sigma_f/\Sigma_a$)

Because we have included both fast and thermal neutrons in this formalism, it is general and the multiplication factor $k = p\epsilon\eta f$, we obtain is the standard infinite medium multiplication factor quoted in the text books for a homogeneous system.

$$k = p\epsilon\eta f \tag{7.16}$$

The above statement for k is called the four-factor formula.

The criticality factor for a finite reactor, can be expressed as

$$C = k\, P_t\, P_f = kP \tag{7.17}$$

$$P = P_t\, P_f$$

where P_t and P_f are, respectively, the non-leakage probabilities for thermal and fast neutrons. The particular mathematical expressions for P_t and P_f depend upon the type of reactor one is concerned with and the specific theoretical model used to compute these probabilities.

Using the relation $L^2 = D/\Sigma_a$ and the four-factor formula, the balance equation can be written as

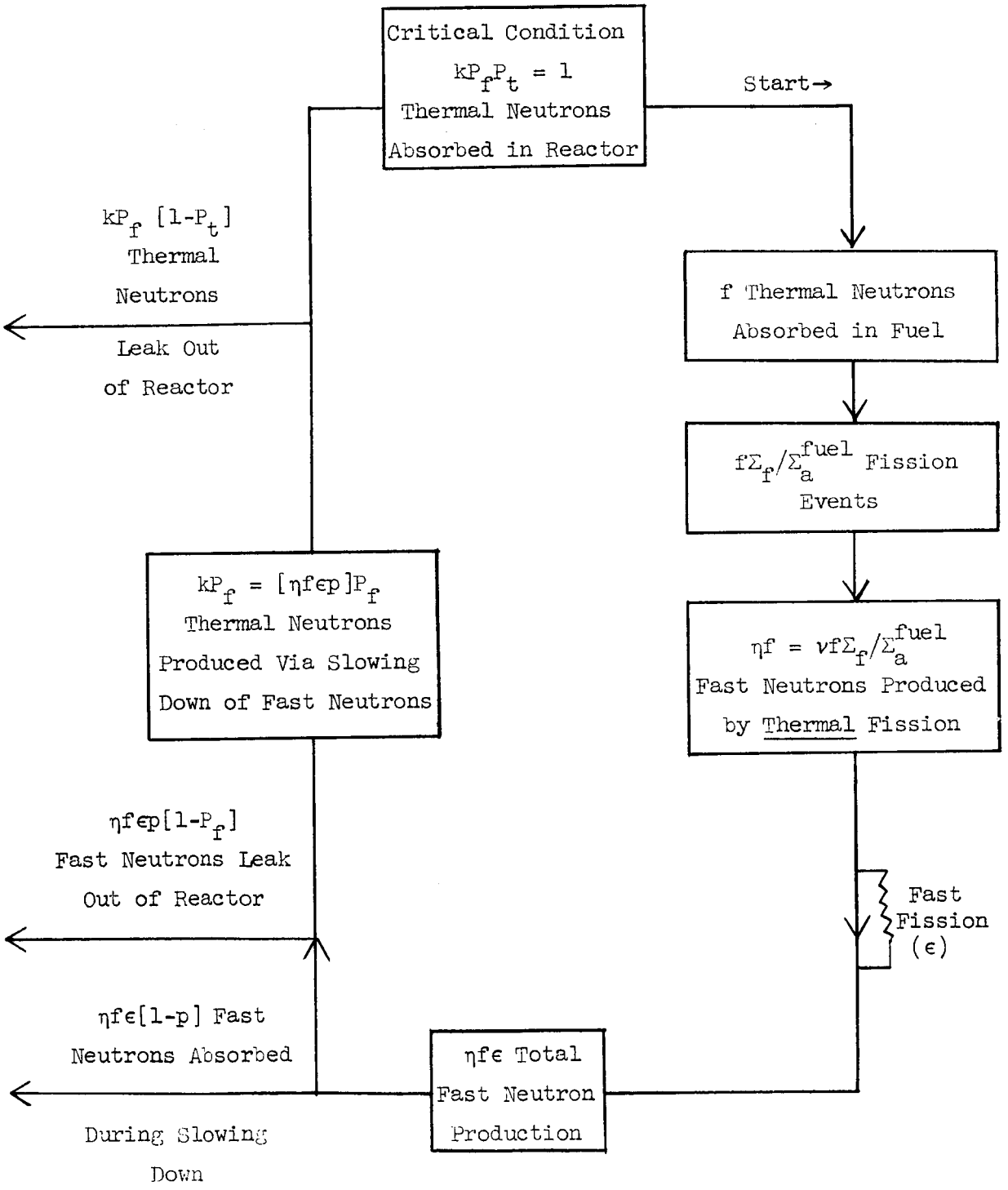
$$\nabla^2\Phi + \frac{k - 1}{L^2}\ \Phi = 0 \qquad (7.18)$$

and B_m^2 as

$$B_m^2 = \frac{k - 1}{L^2} \qquad (7.19)$$

Figure 7.1 is a schematic representing the neutron cycle for a critical thermal reactor.

Fig. 7.1 Neutron Cycle for Critical Thermal Reactor (GE, 202)



8. BARE HOMOGENEOUS REACTOR (WW, 378-412); (GE, 191-224)

The theory of the bare homogeneous reactor is fundamental to all reactor theory. It is relatively easy to use and in the multigroup formulation quite accurate. (In multigroup formalism, the neutron energy range is split up into several subranges, each subrange being called an energy group. Neutrons in each group are treated as if they were monoenergetic so that diffusion theory can be used. Each group receives neutrons slowing down from the group above it and, in turn, supplies neutrons to the group below it. Except for their source terms, the equations for each energy group are the same as the diffusion equation for thermal neutrons we looked at in the previous section). We will treat Bare Homogeneous Reaction theory in terms of the First and Second Fundamental Theorems of Reactor Theory, which will be stated without proof.

8.1 FIRST FUNDAMENTAL THEOREM OF REACTOR THEORY

First Fundamental Theorem:

1. The stationary neutron flux $\Phi(\vec{r}, E)$ in a critical bare reactor is separable in space and energy.

$$\Phi(\vec{r}, E) = \varphi(E)\,\Psi(\vec{r}) \tag{8.1}$$

2. The space distribution of the flux $\Psi(\vec{r})$ is the fundamental solution to the wave equation

$$\nabla^2\Psi + B_g^2\Psi = 0 \tag{8.2}$$

By fundamental solution is meant a function which is positive throughout the reactor and which vanishes on the extrapolated boundary of the reactor. The extrapolation distance d is 0.7λ, according to transport theory, where λ is the neutron mean free path. The constant B_g^2 in the equation for Ψ is determined entirely by the size and shape of the reactor. It is called the geometrical buckling.

The design of a bare critical reactor amounts to achieving equality between B_m^2 for the chosen material composition and B_g^2 for the specific geometry concerned.

8.2 ALL THERMAL REACTOR (GENERAL)

Just as we started the discussion of infinite reactors with a computation for the all-thermal or one-group approximation we will also begin the discussion of finite reactors with a simple one-group treatment. Because all neutrons have the same energy E, $\Phi(\underline{r}, E) =$ constant x $\Psi(\underline{r}) = \Phi(\underline{r})$. At a given point $\underline{r}$ in the reactor we have that, from the standpoint of reactor composition, a reactor is critical if $D\nabla^2 \Phi + (\nu\Sigma_f - \Sigma_a)\Phi = 0$, i.e.,

$$\nabla^2 \Phi + \frac{k-1}{L^2}\Phi = 0$$

The First Fundamental Theorem states that, from a geometrical standpoint, a reactor if $\nabla^2 \Phi + B_g^2 \Phi = 0$.

Hence for the critical state it follows that

$$B_g^2 = \frac{k-1}{L^2} \tag{8.3}$$

We must now stop a moment to introduce the static criticality factor (approximation) and determine its value for a bare, all-thermal reactor. The idea of the static criticality factor is based on the following approximation which is valid if the criticality factor is close to unity, i.e., $0.9 < C < 1.1$. In the actual reactor

$$C = kP = \nu \left[\frac{\Sigma_f}{\Sigma_a^{fuel}}\right]\left[\frac{\Sigma_a^{fuel}}{\Sigma_a}\right]P \tag{8.4}$$

where P is the neutron non-leakage probability. One defines the number ν' to be that value of ν which gives

$$C' = \left[\nu' \frac{\Sigma_f}{\Sigma_a^{fuel}} \frac{\Sigma_a^{fuel}}{\Sigma_a} P\right] = 1 \tag{8.5}$$

In other words, ν' is the particular value of ν required to make the reactor critical. The static criticality factor is defined as

$$C = \frac{\nu}{\nu'} \tag{8.6}$$

If $\nu' > \nu$, C is less than one and the reactor is subcritical. If $\nu' < \nu$, C is greater than one and the reactor is supercritical.

On this basis, the base reactor would be critical if k had the value k' given by

$$\frac{k' - 1}{L^2} = B_g^2 \tag{8.7}$$

where $k' = \nu' \Sigma_f/\Sigma_a$, i.e., $\nu' = \Sigma_a k' /\Sigma_f$.

Solving for k' and ν' on obtains

$$k' = 1 + B_g^2 L^2 \tag{8.8}$$

and

$$\nu' = \Sigma_a (1 + B_g^2 L^2)/\Sigma_f \tag{8.9}$$

By the definition of k,

$$\nu = \Sigma_a k/\Sigma_f \tag{8.10}$$

Now, having expressions for the numbers ν' and ν, the static criticality factor can be written as

$$C = \frac{\nu}{\nu'} = \frac{k}{1 + B_g^2 L^2} \tag{8.11}$$

In general, from the neutron economy structure,

$$C = kP \tag{8.12}$$

hence, it follows at once that the non-leakage probability P for an all-thermal bare reactor is

$$P = \frac{1}{1 + B_g^2 L^2} \tag{8.13}$$

This result always holds for the thermal neutron group, independently of the manner chosen to represent fast neutron behavior.

8.3 ALL-THERMAL SLAB REACTOR (COMPUTATIONAL EXAMPLE)

The ideas listed above can be somewhat fixed in one's head by carrying through a simple example. Consider a slab reactor of physical thickness h. (See Figure 8.1). Let x = 0 be the center plane of the slab. According to the First Fundamental

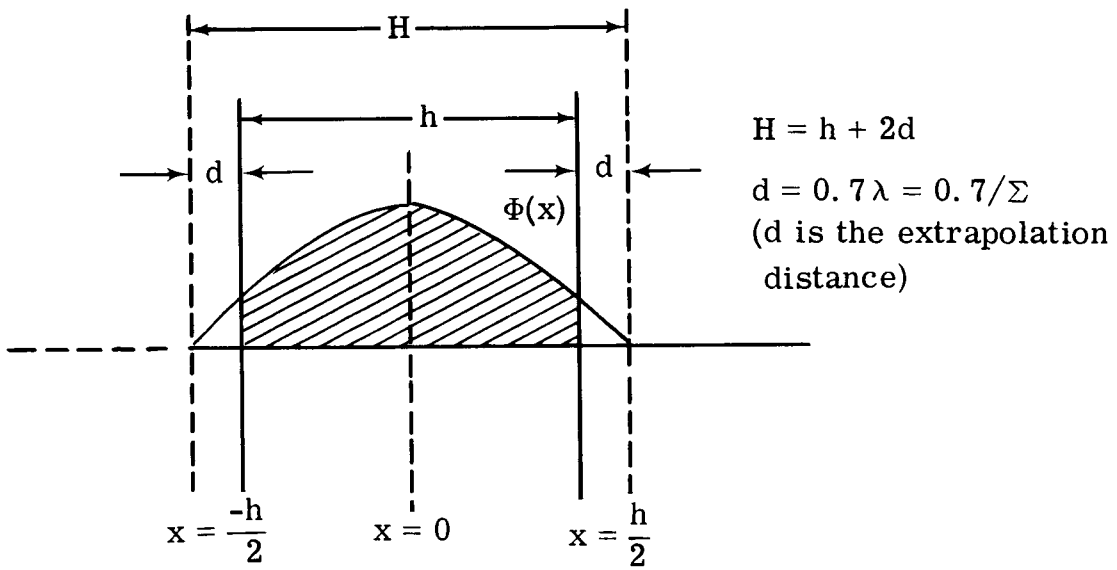


Fig. 8.1

Theorem, Φ vanishes at $x = \pm(H/2)$ and is positive on the interval $-H/2 < x < H/2$. The fundamental solution to

$$\nabla^2 \Phi + B_g^2 \Phi = 0 \tag{8.14}$$

gives

$$B_g^2 = (\pi/H)^2 \quad \text{Slab Reactor} \tag{8.15}$$

The fundamental solution, itself, being

$$\Phi(x) = K\cos(\pi x/H) \tag{8.16}$$

The constant of integration K is determined by the power level of the reactor. The power produced by a slab reactor is infinite, however, if one considers the power P generated in a column of length h through the

slab with a base area of 1 cm^2 and with its longitudinal axis normal to the slab faces, the constant K can be evaluated.

The number of fission events per second in this column is the integral of $\Sigma_f \Phi$ over the physical thickness of the slab.

$$\text{Fission rate} = K\Sigma_f \int_{-h/2}^{h/2} \cos(\pi x/H) \tag{8.17}$$

$$= 2K\Sigma_f \left(\frac{H}{\pi}\right) \sin\left(\frac{\pi h}{2H}\right) \tag{8.18}$$

Because 3.1×10^{10} fission per sec constitute a power production of 1 watt, it follows

$$P = \frac{2K\Sigma_f}{3.1}\left(\frac{H}{\pi}\right) \times 10^{-10} \times \sin\left(\frac{\pi h}{2H}\right) \text{(watts)} \tag{8.19}$$

If P were one watt, then K would be

$$K = [3.1\pi \times 10^{10}]/[2\Sigma_f H \sin(\pi h/2H)] \tag{8.20}$$

The power distribution for a reactor is:

$$\text{Power Dist.} = 200 \cdot \Sigma_f \Phi \text{ meV/cm}^3\text{-sec}$$

$$= \frac{\Sigma_f \Phi}{3.1 \times 10^{10}} \text{ watts cm}^3 \tag{8.21}$$

In a thermal reactor, therefore, the shape of the power distribution is the same as that of the thermal flux.

Problem: Compute the current of escaping neutrons at the faces of a critical slab reactor.

Hint: $J = \frac{D}{\lambda} \text{grad } \Phi$, $J_x = \frac{D}{\lambda} \partial \Phi / \partial x$.

Problem: Compute the average flux in a critical slab reactor.

Hint: $\langle \Phi \rangle = \int_{-h/2}^{h/2} \Phi \, dx/h$

Problem: Given P = 0.1 megawatt, H = 142 cm, $\Sigma = 0.8$; compute Φ_{max} for a critical slab reactor.

Hint: Remember the extrapolation distance d.

8.4 RECTANGULAR PARALLELEPIPED

Let a, b, c be the physical dimensions of a parallelepiped reactor, along the x, y and z axes respectively, with the origin at the center of the reactor. Let $A' = a + 2d$, $B' = b + 2d$ and $C' = c + 2d$ be the extrapolated dimensions.

In this case

$$B_g^2 = \left(\frac{\pi}{A'}\right)^2 + \left(\frac{\pi}{B'}\right)^2 + \left(\frac{\pi}{C'}\right)^2 \tag{8.22}$$

$$\Phi(x, y, z) = K \cos\left(\frac{\pi x}{A'}\right) \cos\left(\frac{\pi y}{B'}\right) \cos\left(\frac{\pi z}{C'}\right) \tag{8.23}$$

The constant K is again determined by the power level of the reactor.

8.5 SPHERICAL REACTOR

Let r_o be the physical radius of the reactor and $R = r_o + d$ the extrapolated radius, then

$$B_g^2 = \left(\frac{\pi}{R}\right)^2 \tag{8.24}$$

$$\Phi(r) = \frac{K}{r} \sin\left(\frac{\pi r}{R}\right) \tag{8.25}$$

The constant K is determined by the power level of the reactor.

8.6 CYLINDRICAL REACTOR

Let r_o be the physical radius of the cylinder and h its length. Denote the extrapolated radius by $R = r_o + d$ and the extrapolated length by $H = h + 2d$ and take the origin of the coordinate system at the center of the cylinder. B_g^2 and $\Phi(r, z)$ are then

$$B_g^2 = \left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2 \tag{8.26}$$

$$\Phi(r, z) = K J_o\left(\frac{2.405 r}{R}\right) \cos\left(\frac{\pi z}{H}\right) \tag{8.27}$$

J_o the zero order Bessel function of the first kind.

8.7 MINIMUM CRITICAL VOLUME

Reactors are usually designed to be critical at the smallest possible volume for reasons of economy if nothing else. This is called the minimum critical volume. Table 8.1 lists minimum critical volumes for three reactor shapes. Note that,

TABLE 8.1

MINIMUM CRITICAL VOLUME

Shape	Minimum Critical Volume
Parallelepiped	$161/B_m^3$
Cylinder	$148/B_m^3$
Sphere	$130/B_m^3$

for a specified composition, i.e., material buckling B_m^2, the minimum critical volume for a sphere is less than that for any other shape.

8.8 SUMMARY OF DIFFUSION THEORY FORMALISM (MONOENERGETIC)

$D\nabla^2\Phi$ = number of neutrons gained per unit volume per unit time by diffusion

$\kappa = \sqrt{\Sigma_a/D}$ (Inverse Diffusion Length)

$L = \sqrt{D/\Sigma_a}$ (Diffusion Length)

$$\frac{\partial n}{\partial t} = \underbrace{D\nabla^2\Phi + S}_{\text{Gain}} - \underbrace{\Sigma_a\Phi}_{\substack{\text{Absorption} \\ \text{Loss}}} \quad \text{(Neutron Balance Equation)}$$

$\vec{J} = -D\vec{\nabla}\Phi$ (Diffusion Current)

$$\Phi(r) = \frac{e^{-r/L}}{4\pi Dr} \quad \text{(Diffusion Kernel for Unit Point Source)}$$

$\langle r^2 \rangle = 6L^2$ (Mean Square Diffusion Distance - three dimensions)

$$\Phi(x) = \frac{e^{-x/L}}{2D\kappa} \quad \text{(Diffusion Kernel for Unit Plane Source)}$$

~~$B_g^2 = (\pi/H^2)$ Slab~~

$B_g^2 = (\pi/H)^2$ Slab

$B_g^2 = (\pi/a)^2 + (\pi/b)^2 + (\pi/c^2)$ Rectangular Parallelepiped

$B_g^2 = (\pi/R)^2$ Sphere

$B_g^2 = (2.405/R)^2 + (\pi/H)^2$ Cylinder (R, H)

Dynamic Criticality Factor:

$$C_d = \frac{\text{Neutron Production Rate}}{\text{Neutron Loss Rate}}$$

Static Criticality Factor:

$$C_s = \frac{\nu}{\nu'}$$

ν = actual number of fission neutrons produced per fission

ν' = required number of fission neutrons produced per fission so that C = 1

Neutron Number Density n

n = number of neutrons per unit volume

n(E)dE = number of neutrons per unit volume with energy in the range dE at E

n(E) = number of neutrons per unit volume per unit energy at E

$$n = \int_0^\infty n(E)dE$$

$n(\vec{r})$ = number of neutrons per unit volume at $\vec{r}$

$n(\vec{r}, E)\, dE d\vec{r}$ = number of neutrons in the volume element $d\vec{r}$ at $\vec{r}$ with energy in the range dE at E

$n(\vec{r}, E)$ = number of neutrons per unit volume per unit energy at $\vec{r}$ and E

$n(\vec{r}) = \int n(\vec{r}, E)\, dE$

$n(E) = \int n(\vec{r}, E)\, d\vec{r}$

$n = \int d\vec{r} \int n(\vec{r}, E)\, dE$

$n(\vec{r}, E, \vec{\Omega})$ = number of neutrons per unit volume per unit energy per unit solid angle at $\vec{r}$, $\vec{E}$ and velocity in direction $\vec{\Omega}$

$n = \int d\vec{r} \int d\vec{\Omega} \int n(\vec{r}, E, \vec{\Omega})\, dE$

Scalar Neutron Flux Φ

$\Phi(\vec{r}, E) = n(\vec{r}, E)\, v(E)$ neutrons per cm^2-sec, per unit energy at $\vec{r}$, where v is the neutron velocity magnitude associated with energy E, i.e., $v = \sqrt{2E/m_n}$.

$\Phi(\vec{r}, E)$ = track length traced out per unit volume per unit energy per second at $\vec{r}$ by neutrons with energy E.

$$\Phi(E) = \int \Phi(\vec{r}, E)\, d\vec{r}$$

$$\Phi(\vec{r}) = \int \Phi(\vec{r}, E)\, dE$$

$$\Phi = \int d\vec{r} \int \Phi(\vec{r}, E)\, dE$$

Macroscopic Cross Section Σ

$\Sigma = N\sigma$ (N = number of nuclei per cm^3)

$\Sigma(E)$ = probability per unit of track length for a neutron of energy E to react with nuclei

$\Sigma(E)\,\Phi(E)$ = number of reactions per unit energy per unit volume per second undergone by neutrons with energy E

$\Sigma(E)\,\Phi(\vec{r}, E)$ = number of reactions per unit volume per unit energy per second undergone by neutrons with energy E at $\vec{r}$

$\int \Sigma(E)\,\Phi(\vec{r}, E)\, dE$ = number of neutron reactions per unit volume per second at $\vec{r}$

$\int \Sigma(E)\,\Phi(\vec{r}, E)\, d\vec{r}$ = number of reactions per unit energy undergone by neutrons with energy E

$\int d\vec{r} \int \Sigma(E)\,\Phi(E, \vec{r})\, dE$ = total number of neutron reactions per second

Σ_s = scattering reaction cross section

Σ_a = absorption reaction cross section

Σ_f = fission reaction cross section

DB^2 = leakage cross section (diffusion theory)

9. FAST NEUTRON LEAKAGE

9.1 SECOND FUNDAMENTAL THEOREM

The fast neutron non-leakage probability P_f is specified by the Second Fundamental Theorem of Reactor Theory.

Second Fundamental Theorem:

The non-leakage probability for fast neutrons during moderation in a uniform bare reactor* is the Fourier transform of the slowing down kernel K(r).

$$P_f = 4\pi \int_0^\infty K(r) \left(\frac{\sin B_g r}{B_g r} \right) r^2 \, dr \tag{9.1}$$

*Note that nothing is said aboug the reactor having to be critical in the Second Fundamental Theorem.

The slowing down kernel K(r) is defined as the number of thermal neutrons created per unit volume at position r in an infinite system, given one fission neutron at the origin and no absorption. The integral of K(r) over all space is unity.

$$\int_0^\infty 4\pi r^2 \, K(r) \, dr = 1 \tag{9.2}$$

Slowing down kernels can be theoretically derived, but in practice, synthetic slowing down kernels, obtained by making analytical fits to experimental data, are usually adopted. A restricted description of the neutron slowing down process and the use of slowing down kernels follows immediately.

9.2 NEUTRON SLOWING DOWN

The most important questions concerning neutron slowing down are:

1. How far does the neutron travel during the slowing down process? This distance is represented by the quantity τ which is defined to be one-sixth the average square distance between the fast neutron origin and the point at which it becomes thermalized. τ determines the fast neutron non-leakage probability.
2. How many fast neutrons avoid absorption while slowing down? As mentioned before, this quantity is represented by the fraction p. p is called the resonance escape probability in honor of the first reactors which used natural uranium fuel.

We will assume that fast neutrons slow down via elastic collisions with moderator nuclei (and reflector nuclei in the case of reflected reactors). The magnitude of the fractional neutron energy loss per collision is an extremely important quantity. It determines how many collisions are required to thermalize fast neutrons and this number determines both the type of slowing down kernel one should use and, together with the absorption cross section, the value of the absorption escape probability p.

When a neutron scatters elastically with a nucleus having a mass number A, the largest fraction of its energy that can be transferred to the nucleus is $(1 - \alpha)$, where

$$\alpha = \left(\frac{A-1}{A+1}\right)^2 \tag{9.3}$$

(This is not the α used in (WW). It is the α used in (GE).) Notice that the lighter the nucleus the smaller is α and hence the larger the maximum fractional energy loss. This fact is illustrated by the following table.

Nucleus	A	α	$(1 - \alpha)$	ξ
H	1	0	1	1.000
Be	9	0.640	0.360	0.209
C	12	0.716	0.284	0.158
O	16	0.778	0.222	0.120
Fe	56	0.932	0.068	0.0353
U	235	0.984	0.016	0.00838

The quantity ξ is the average logarithmic energy decrement per collision, E_b and E_a being the

$$\xi = \langle \log_e (E_b/E_a) \rangle \tag{9.4}$$

neutron energy before and after collision, respectively.

ξ is useful because for a given moderator nucleus it is a constant in contrast to the absolute amount of energy lost which decreases with neutron energy. This being the case, the average number of collisions required to slow a neutron from energy E_1 to energy $E_2 < E_1$ is

$$\left\{\begin{matrix}\text{Ave. Number Collision} \\ \text{from } E_1 \text{ to } E_2\end{matrix}\right\} = \frac{\log_e(E_1/E_2)}{\xi} \tag{9.5}$$

Because the average fission neutron energy is 2 meV it is useful to remember that

$$\begin{matrix}\text{Ave. Number of Collisions to} \\ \text{Thermalize a 2 meV neutron}\end{matrix} = \frac{\log_e(2 \times 10^6/.0254)}{\xi} = \frac{18.2}{\xi} \tag{9.6}$$

Many authors use the neutron <u>lethargy</u>, $u = \log_e (10^7/E)$, rather than the energy in slowing down calculations. (Given u, the associated energy is $E = 10^7\ e^{-u}$ eV). The reason for this formulation will soon be evident.

Problem: Verify that $du = -dE/E$ and $u_2 - u_1 = \log_e (E_1/E_2)$.

If the cross sections were energy independent, the probability p for slowing down without absorption would be

$$p \cong \left(\frac{\Sigma_s}{\Sigma}\right)^{18.2/\xi} = \left(1 - \frac{\Sigma_a}{\Sigma}\right)^{18.2/\xi} \tag{9.7}$$

because Σ_s/Σ is the probability to scatter rather than suffer absorption in any given single collision and an average of $18.2/\xi$ "successful" scattering collisions are required for thermalization. Cross sections are energy dependent, however, so the simple constant cross section scheme is not valid as it stands. It can be used, though, to approximate p by dividing the energy range from 0.0254 eV to 2×10^6 eV into a succession of non-overlapping energy intervals ΔE_i, with lower bound E_i, in each of which the cross sections are almost constant. The probability to get through the i-th interval is then

$$p_i = \left(\frac{\Sigma_s}{\Sigma}\right)_i^{\left(\log_e \frac{E_i+\Delta E_i}{E_i}\right)/\xi} \cong \left(\frac{\Sigma_s}{\Sigma}\right)_i^{\frac{\Delta E_i}{\xi E_i}} \tag{9.8}$$

Note: $\log_e \left(\frac{E_i + \Delta E_i}{E_i}\right) = \log_e \left(1 + \frac{\Delta E_i}{E_i}\right) \cong \frac{\Delta E_i}{E_i}$ if $\Delta E_i << E_i$

The probability to run the whole gamut of energy intervals is p, by definition, and is given by

$$p = \prod_i p_i = \prod_i \left(\frac{\Sigma_s}{\Sigma}\right)_i^{E_i/\xi E_i} \tag{9.9}$$

Because the logarithim of a product is the sum of the logarithim of the factors is valued, one has,

$$\log p = \sum_i \frac{\Delta E_i}{\xi E_i} \log \left(\frac{\Sigma_s}{\Sigma}\right)_i \tag{9.10}$$

which can be approximated by the integral

$$\log p = \int_{0.0254}^{2x10^6} \left[\log \left(\frac{\Sigma_s}{\Sigma}\right)\right] \frac{dE}{\xi E} = \int_o^{u_t} \log \left(\frac{\Sigma_s}{\Sigma}\right) \frac{du}{\xi} \left(du = -\frac{dE}{E}\right) \tag{9.11}$$

If Σ_a is very small compared with Σ,

$$p = \exp \left[-\int_{0.0254}^{2x10^6} \frac{\Sigma_a}{\Sigma} \frac{dE}{\xi E}\right] = \exp \left[-\int_o^{u_t} \frac{\Sigma_a}{\Sigma} \frac{du}{\xi}\right] \tag{9.12}$$

Note: $\log_e \left(\frac{\Sigma_s}{\Sigma}\right) = \log \left(1 - \frac{\Sigma_a}{\Sigma}\right) \cong -\frac{\Sigma_a}{\Sigma}$, if $\Sigma_a << \Sigma$

The lethargy variable language was used in the expression for p in addition to the energy variable language because it is frequently written this way in texts and the open literature. In lethargy language, ξ is called the average lethargy increase per collision.

9.3 SLOWING DOWN KERNELS

According to the Second Fundamental Theorem

$$P_f = \int_0^\infty 4\pi r^2 K(r) \frac{\sin B_g r}{B_g r} \, dr \tag{9.13}$$

By using only the normalization condition

$$\int_0^\infty K(r) \, 4\pi r^2 \, dr = 1 \tag{9.14}$$

and no other property of K(r), a great deal can be learned about P_f. This is done by expanding $(\sin B_g r)/B_g r$ in an infinite series and integrating term by term.

a. If the reactor is infinite, i.e., $B_g = 0$, then

$$P_f = \int_0^\infty K(r) \, 4\pi r^2 \, dr = 1 \tag{9.15}$$

$$\text{as } \lim_{x \to 0} \frac{\sin x}{x} = 1$$

b. Now suppose the reactor is finite but large enough that $B_g << 1$ so that the integral of the infinite series

$$\frac{\sin B_g r}{B_g r} = 1 - \frac{1}{6} B_g^2 r^2 + \frac{1}{120} B^4 r^4 - \frac{1}{5040} B^6 r^6 + \ldots \tag{9.16}$$

can be broken off at the squared term. When this can be done,

$$P_f \cong \int_0^\infty 4\pi r^2 \, K(r) \left[1 - \frac{1}{6} B_g^2 r^2\right] dr = 1 - \frac{1}{6} B^2 \langle r^2 \rangle = 1 - \tau B^2 \tag{9.17}$$

(Recall that $\tau = \frac{1}{6} \langle r^2 \rangle$ is the slowing down area.) In this case the criticality is

$$C = \frac{k \, (1 - \tau B^2)}{(1 + L^2 B^2)} \cong \frac{k}{(1 + L^2 B^2)(1 + \tau B^2)} \cong \frac{k}{1 + B^2 M^2} \tag{9.18}$$

where $M^2 = L^2 + \tau$ and is called the migration area. M^2 is one-sixth the mean square distance a neutron travels during slowing down, as a fast neutron, and during diffusion after thermalization. Hence, in large reactors, the critical buckling is closely approximated as

$$B^2 = \frac{k - 1}{M^2} \tag{9.19}$$

Fermi Age (Gaussian) Kernel: The age of a neutron of energy E is defined as

$$\tau(E) = \int_E^{E_0} \frac{D}{\Sigma_s} \frac{dE}{\xi E} \tag{9.20}$$

In lethargy language

$$\tau(u) = \int_{u_0}^{u} \frac{D}{\Sigma_s} \frac{du}{\xi} \tag{9.21}$$

Physically, τ is one-sixth the mean square distance a neutron travels in an infinite medium while slowing down from the source energy E_0 to energy E. Age theory is based on the assumption of continuous slowing down, which amounts to assuming $\xi << 1$. It gives

$$K(r) = \frac{e^{-r^2/4\tau}}{(4\pi\tau)^{3/2}} \tag{9.22}$$

$$P_f = e^{-B_g^2 \tau} \tag{9.23}$$

and the criticality equation

$$C = \frac{k e^{-B_g^2 \tau}}{(1 + L^2 B_g^2)} \tag{9.24}$$

$$\cong \frac{k}{(1 + L^2 B_g^2)(1 + B_g^2 \tau)} \quad \text{(large reactor)}$$

This result is in accord with the general result derived above. Age theory is a good approximation for graphite moderated reactors, in fact it was invented for the analysis of large graphite reactors. If fast neutrons are

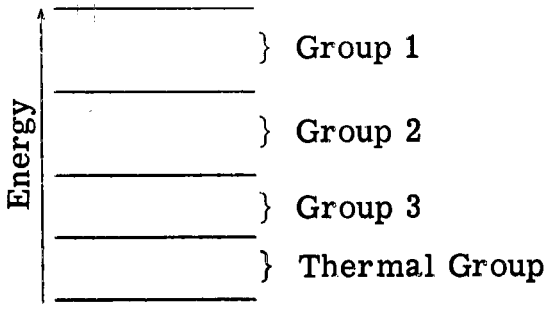
moderated by Be, age theory is less successful as $\xi_{Be} = 0.209$ while $\xi_C = 0.158$. Age theory is no good at all for water moderated reactors because $\xi_H = 1.0$ and slowing down can be accomplished in just a single collision between a neutron and a hydrogen nucleus.

Diffusion Kernel

A good approximation method for neutron slowing down by light nuclei can be obtained by the convolution of several diffusion kernels. A diffusion kernel has the form

$$K(r) = \frac{e^{-r/L}}{4\pi L^2 r} \qquad (9.25)$$

where L is the diffusion length. It represents the number of neutrons per unit volume at a distance r from a unit point source as given by diffusion theory. In this treatment the fast neutron energy range is divided into several subintervals called energy groups. All neutrons in a given group are taken to be monoenergetic, their energy being the average energy for the group. A different diffusion kernel is then assigned to each group. For sake of definiteness suppose three fast neutron groups with kernels $K_1(r)$, $K_2(r)$ and $K_3(r)$ are selected along with the thermal group or level. This is a four group model. Let subscript 1 indicate the highest energy group, 2 the intermediate fast neutron group and 3 the lowest fast neutron group. L_1^2, L_2^2, and L_3^2 will be the diffusion areas for the fast groups and L^2 that for the thermal group. L_i^2 is one-sixth the mean square



distance traveled by neutrons in the i-th group before they slow down into the next lower group.

The probability to slow down through all three fast groups into the thermal group is

$$P_f = P_{1f} \times P_{2f} \times P_{3f} \qquad (9.26)$$

where

$$P_{if} = \int_0^\infty 4\pi r^2 K_i(r) \frac{\sin B_g r}{B_g r} \, dr \tag{9.27}$$

In the case of diffusion kernels these integrals are

$$P_{if} = \frac{1}{(1 + L_i^2 B^2)} \tag{9.28}$$

Hence

$$P_f = \left(\frac{1}{1 + L_1 B^2}\right)\left(\frac{1}{1 + L_2 B^2}\right)\left(\frac{1}{1 + L_3 B^2}\right) \tag{9.29}$$

and

$$P = P_t P_f = \frac{P_f}{(1 + L^2 B^2)} \tag{9.30}$$

This approach (convolution of slowing down kernels) is similar to the probability model used in Section 9.2 to compute p. The diffusion kernel criticality equation is

$$C = \frac{k}{(1 + L_1^2 B^2)(1 + L_2^2 B^2)(1 + L_3^2 B^2)(1 + L^2 B^2)} \tag{9.31}$$

An example of the accuracy of the four group method for a U^{235} water-moderated, base cylindrical reactor is given by Figures 9.1 and 9.2. In this example synthetic diffusion kernels were used where L_1 = 4.49 cm, L_2 = 2.05 cm and L_3 = 1.0 cm. L = 2.88 cm was used for the thermal group. Figure 9.1 compares critical height calculations for (1) Age Theory (b) Two-Group and (c) Four-Group with experiment. Age Theory is clearly not suited for water moderated reactors. The four-group calculation is quite accurate. In constructing synthetic diffusion kernels, the analytical fits are made so that the resulting kernels give the experimental slowing down area and the same number of thermal neutrons as experiment. The thermal neutron slowing down density given by each method is given in Figure 9.2.

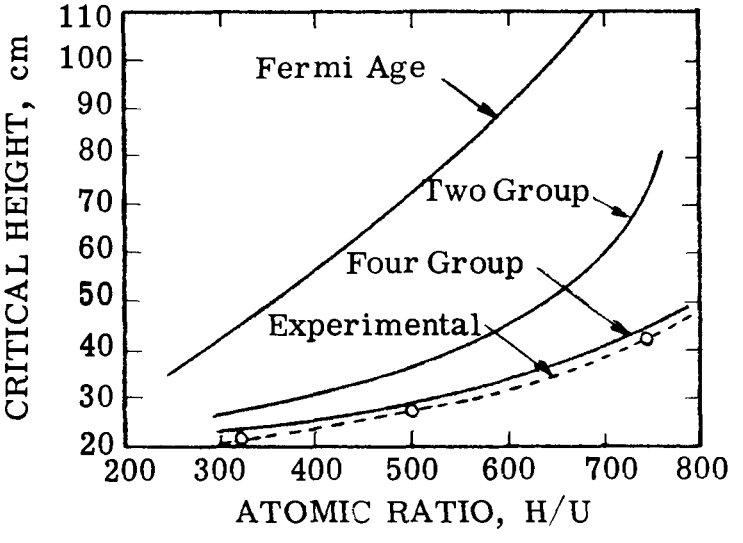


Fig. 9.1 - Comparison of critical height calculations for a bare, finite cylindrical reactor. The moderator is water. H refers to hydrogen and U to the uranium fuel

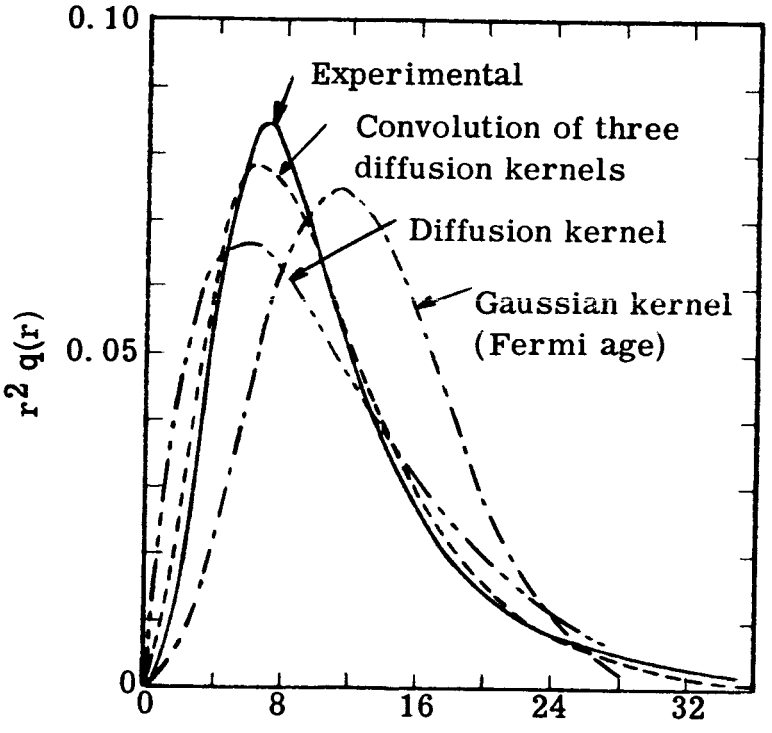


Fig. 9.2 - Experimental and calculated slowing down density, q(r), in water. q(r) is the number of thermal neutrons created per unit volume per second at a distance r from a unit source in this instance.

9.4 SOLUTION OF A TYPICAL PROBLEM

Problem: A homogeneous mixture of 1 part by atoms of U^{235} to 15000 parts of graphite is to be used to construct a bare spherical reactor.
Calculate (a) critical size
(b) critical mass

Assume the following constants:
$\nu = 2.5$
$\sigma_a (U^{235}) = 681$ barns (σ_a^u)
$\sigma_f (U^{235}) = 580$ barns (σ_f^u)
τ (Graphite) $= 350\ cm^2$
L^2 (Graphite) $= 2580\ cm^2$
ρ (Graphite) $= 1.6\ gm/cm^3$ (ρ^c)
σ_a (Graphite) $= 0.003$ barns (σ_a^c)

Solution: $A_w^u = 235$ g/mol $\quad A_w^c = 12$ g/mol

The calculation of k is simplified since in the absence of U^{238}, we assume $\epsilon = p = 1$. Therefore

$$k = \eta f = \nu \frac{\sigma_f^u}{\sigma_a^u} \cdot \frac{\Sigma_a^u}{\Sigma_a^u + \Sigma_a^c}$$

$$= 2.5 \frac{580}{681} \quad \frac{1}{1 + \frac{N^c}{N^u} \frac{\sigma_a^c}{\sigma_a^u}}$$

where N^c is the number of atoms of carbon per cm^3 of the mixture and N^u is the corresponding number for uranium atoms. Thus

$$k = 2.5 \frac{580}{681} \quad \frac{1}{1 + 15000 \frac{0.003}{681}} = 2.00$$

The Fermi Age is a function of Σ_s, D and ξ. It is simple to show that for low concentrations of heavy nuclei in light moderators, τ is not significantly different from that of the pure moderator. Hence τ for the reactor is equal to τ (graphite) = 350 cm^2.

The same cannot be said however for $L^2 \equiv D/\Sigma_a$. Although D, the diffusion coefficient is not affected, the large absorption coefficient of uranium compared with graphite makes it important even in small concentrations.

Since L^2 (Graphite) $= \frac{D}{\Sigma_a^c}$

and L^2 (reactor) $= \frac{D}{\Sigma_a^c + \Sigma_a^u}$

it follows that

$$L^2 \text{ (reactor)} = \frac{L^2 \text{ (graphite)}}{1 + \frac{\Sigma_a^u}{\Sigma_a^c}}$$

$$= \frac{2580}{1 + \frac{681}{0.003 \times 15000}} = \underline{160.0 \text{ cm}^2} .$$

Now, applying equation (9.24)

$$\frac{2.00 \times e^{-350 B_g^2}}{1 + 160.0\, B_g^2} = 1$$

and therefore, $B_g^2 = 0.0014 = \left(\frac{\pi}{R}\right)^2$

and $\underline{\underline{R = 83.7 \text{ cm.}}}$

The slight dilution of uranium will not affect the atomic density of carbon. Therefore

$$N^c = \frac{\text{Avogadro's number} \times \rho \text{ (graphite)}}{\text{Atomic weight}} = \frac{6.025 \times 10^{23}\, \rho^c}{A_w^c}$$

Therefore, since

$$N^u = \frac{N^c}{15000} = \frac{6.025 \times 10^{23}\, m^u}{A_w^u}$$

where m^u = mass of uranium per cm^3, it follows

$$m^u = \frac{\rho^c A^u}{15000 A^c} = \frac{1.6 \times 235}{15000 \times 12} = 2.09 \times 10^{-3} \text{ grams/cm}^3$$

The critical mass of uranium M_c^u in the sphere is then

$$M_c^u = m_u V = 2.09 \times 10^{-3} \times \frac{4}{3} \pi (83.8)^3 = \underline{\underline{5.15 \text{ Kg}}}$$

10. NON-UNIFORM REACTORS

By a homogeneous reactor is meant a reactor where small-scale composition is uniform and isotropic. All cross sections are independent of position in a homogeneous reactor. In non-uniform reactors the cross sections are position dependent and reactor calculations are more complex than those for uniform reactors, both in the conceptual sense and with regard to the mechanics of computation. All real reactors are to some degree non-uniform. The non-uniformity usually consists of:

1. The presence of a reflector which serves to deflect neutrons back into the fueled core and hence cut down fast neutron leakage.
2. The presence of control rods which serve to regulate the power level of the reactor.
3. Non-uniform fuel loadings which serve to increase the efficiency of the reactor by flattering the power distribution.

The material composition in a non-uniform reactor usually changes abruptly (discontinuously in mathematical language). Hence, at the interface between two different materials the neutron flux has to adjust itself to a compromise status between the two different characters it would exhibit in an infinite system composed purely of either of the two adjacent materials. The fundamental assumption used in the diffusion theory of non-uniform reactor are:

1. The diffusion balance equation can be set up in each different material region and gives the neutron flux for each region.
2. At the boundary between two regions the flux and ~~its derivative~~ the current are continuous.
3. The flux is zero at the extrapolated boundary of all external regions.

If there are a large number of different material regions the matching of the flux and ~~its derivative~~ the current at each boundary becomes a very tedious undertaking.

10.1 REFLECTOR SAVINGS (WW, 495-500)

Figure 10.1 shows the effect of a reflector on the thermal neutron flux. The cross hatched area represents the gain in flux integral instituted by the reflector and it is at once clear that, given a core composition, the dimensions of a critical reflected core are smaller than those of a critical bore core. The difference, δ, is called the reflector savings.

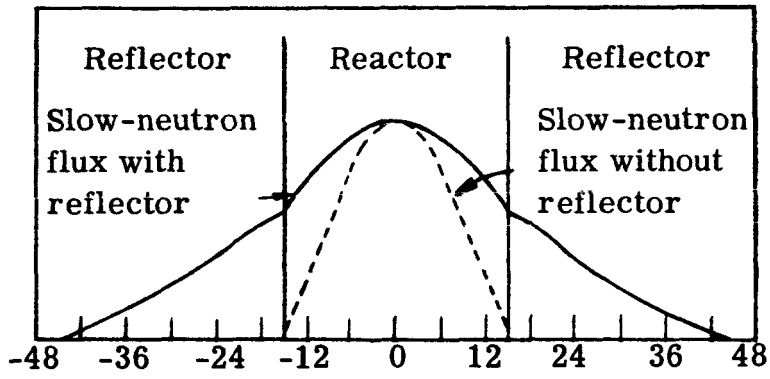


Fig. 10.1 - Slow-neutron flux in spherical U^{235}, water-moderated reactor with and without a beryllium oxide reflector. The maximum-to-average flux ratio in the core with reflector is 1.4; without reflector it is 3.2. The reflector, with extrapolation distance, is 30 cm thick.

Reflected reactor calculations usually begin with an estimate based on the all-thermal model which is then refined in a two-group calculation. If the reflector is thin ($T << L_r$),

$$\delta \cong \frac{D_c}{D_r} T \quad (T = \text{reflector thickness}) \tag{10.1}$$

and if the reflector is thick ($T \cong L_r$),

$$\delta \cong \frac{D_c}{D_r} L_r \quad (L_r = \text{diffusion length in reflector}) \tag{10.2}$$

If R is the core dimension, then the material buckling for critical operation is

$$B_m^2 = \frac{\pi^2}{4\,(R+\delta)^2} \tag{10.3}$$

and the criticality is

$$C = \left[\frac{k}{1 + \frac{\pi^2 L^2}{4(R+\delta)^2}}\right] \tag{10.4}$$

where L is the diffusion length for the core.

The change in criticality, ΔC, obtained placing a reflector characterized by δ on a reactor of dimension R is

$$\Delta C = \left[\frac{k}{1 + \frac{\pi^2 L^2}{4(R+\delta)^2}} - \frac{k}{1 + \frac{\pi^2 L^2}{4R^2}}\right] \cong \frac{k-1}{k}\left[\frac{2\delta}{R}\right] \tag{10.5}$$

if $\delta << R$

10.2 ONE-GROUP CALCULATION FOR REFLECTED SLAB REACTOR (GE, 229-236)

This example will show how a one-group reflector calculation is done. The sub script 'c' refers to the core region and the subscript 'r' to the reflector region. Figure 10.2 describes the geometry for a slab calculation. h is the physical width of the core, T is sum of the physical width of a reflector slab t and the extrapolation distance d_r. The fluxes Φ_c and Φ_r are obtained by solving the coupled differential equation.

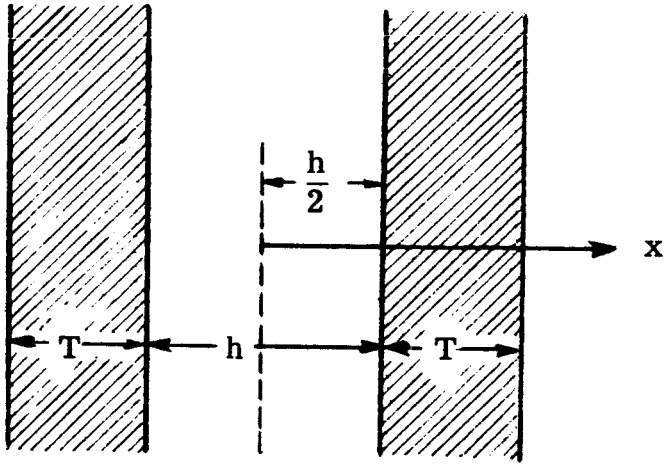


Fig. 10. 2 - Reflected slab reactor geometry

$$\nabla^2 \Phi_c + \frac{k - 1}{L_c^2} \Phi_c = 0 \tag{10.6}$$

$$\nabla^2 \Phi_r - \frac{\Phi_r}{L_r^2} = 0 \tag{10.7}$$

using the boundary conditions

$$\Phi_r \left(\frac{h}{2} + T \right) = 0 \tag{10.8}$$

$$\Phi_c (h/2) = \Phi_r (h/2) \tag{10.9}$$

$$D_c \frac{d\Phi_c}{dx} (h/2) = D_r \frac{d\Phi_r}{dx} (h/2) \tag{10.10}$$

This gives

$$\Phi_c (x) = A \cos B_c x \tag{10.11}$$

$$\Phi_r (x) = C \sinh \kappa_r \left(\frac{h}{2} + T - x \right) \tag{10.12}$$

As before A is determined by the power level of the reactor. The criticality condition is obtained by substituting $\Phi_c (x)$ and $\Phi_r (x)$ from Equation 10.11 and Equation 10.12 into Equations 10.9 and 10.10. This gives two homogeneous equation for the constants A and C. The result is:

$$D_c B_c \tan \frac{B_c h}{2} = D_r \kappa_r \coth \kappa_r T \tag{10.13}$$

A typical slab reactor reflector savings curve is given by Figure 10.3. The equation for δ is obtained in the following way. By definition

$$\delta = \frac{1}{2} (h_o - h) \tag{10.14}$$

where $h_o = \pi / B_c^m$ is the critical thickness for an unreflected core having material buckling B_c^m. This gives h in terms of δ as in Equation 10.15

$$\frac{h}{2} = \frac{\pi}{2B_c^m} - \delta \tag{10.15}$$

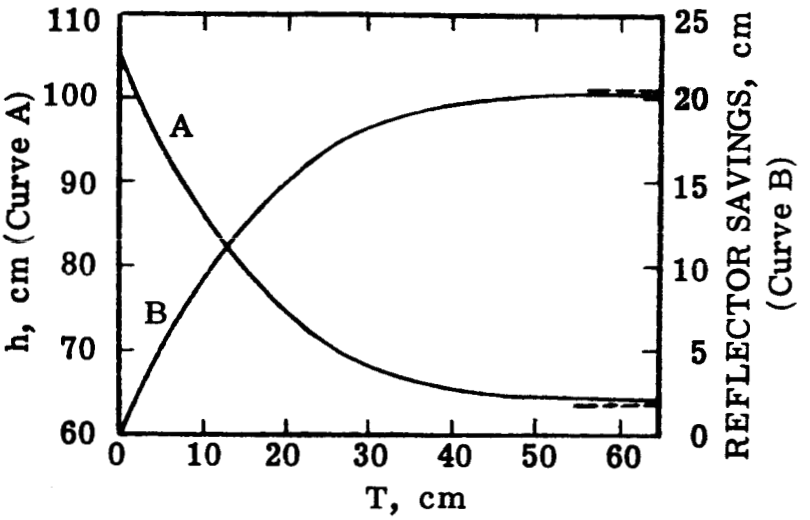


Fig. 10.3 - Reflector savings (Curve B) for a slab reactor. h is slab thickness for a critical reactor (Curve A) and T is the reflector thickness. Note that a 50 cm reflector allows a 40 percent reduction in H, and hence in fuel cost.

Upon substituting Equation 10.15 into Equation 10.13 one obtains

$$\delta = \frac{1}{B_c}\left[\tan^{-1}\left(\frac{D_c B_c}{D_r \kappa_r}\tanh \kappa_r T\right)\right] \qquad (10.16)$$

Problem: Assume a graphite reflector for the U^{235}-graphite reactor of Section 9.4 (a) 10 cm thick (b) 50 cm thick. Since $D_r = D_c$ Equations 10.1 and 10.2 for slab systems also apply to spherical geometry. Compute δ, critical volume and critical fuel mass

a. Because $T = 10 < L_c \cong 50$ Equation 10.1 gives $\delta = 10$ cm and therefore a new critical core radius of $R_c^{ref} = 83.8 - 10 = \underline{73.8 \text{ cm}}$. The critical core volume is

$$V_c^{ref} = \frac{4}{3}\pi (73.8)^3 = \underline{1.68 \times 10^6 \text{ cm}^3}$$

The critical volume of the critical bare core was

$$V_c^{unref} = \frac{4}{3}\pi (83.8)^3 = 2.46 \times 10^6 \text{ cm}^3$$

Hence

$$M_c^u = \frac{1.68 \times 5.15 \text{ kg}}{2.46} = \underline{3.52 \text{ kg}} \quad (10 \text{ cm reflector})$$

b. Because $T = 50 \text{ cm} \cong L_r$ use Equation 10.2 to obtain $\delta = 50$ cm, $R_c^{ref} = 33.8$ cm, $V_c^{ref} = 1.62 \times 10^5 \text{ cm}^3$ and $M_c^u = 0.34$ kg.

10.3 TWO-GROUP CALCULATION FOR A REFLECTED SLAB REFLECTOR

The number of fast neutrons produced per unit volume per sec is $k \Sigma_2 \Phi_{2c}$ where Σ_{2c} is the macroscopic absorption cross section for thermal neutron in the core. This is the fast neutron source term. The balance equation for fast neutrons is therefore

$$D_{1c} \nabla^2 \Phi_{1c} - \Sigma_{1c} \Phi_{1c} + k \Sigma_{2c} \Phi_{2c} = 0 \quad \text{Fast Neutrons (Core)} \qquad (10.17)$$

The thermal neutron source is the product of the fast neutron slowing down cross section Σ_{1c} and the fast flux Φ_{1c} so the balance equation for thermal neutrons is:

$$D_{2c} \nabla^2 \Phi_{2c} - \Sigma_{2c} \Phi_{2c} + \Sigma_{1c} \Phi_{1c} = 0 \quad \text{Thermal Neutron (Core)} \quad (10.18)$$

In the reflector

$$D_{1r} \nabla^2 \Phi_{1r} - \Sigma_{1r} \Phi_{1r} = 0 \quad \text{(no fast neutron source)} \quad (10.19)$$

$$D_{2r} \nabla^2 \Phi_{2r} - \Sigma_{2r} \Phi_{2r} + \Sigma_{1r} \Phi_{1r} = 0 \quad (10.20)$$

We now have four balance equations to solve simultaneously.

Note that all of the balance equations are inhomogeneous excepting the fast flux equation for the reflector. (A differential equation is in homogeneous if it contains functions other than the unknown function one wants to solve for. For example, the inhomogeneous term in the core fast neutron equation is $k\Sigma_{2c} \Phi_{2c}$.) The solution to an inhomogeneous differential equation is equal to the sum of the solution to its homogeneous part and a particular solution specifically geared to the nature of the inhomogeneous part, i.e.,

$$\Phi = \Phi_{homo} + \Phi_{particular}$$

The homogeneous parts of the four balance equations are:

$$\left.\begin{aligned} \nabla^2 \Phi_{1c} + B^2 \Phi_{1c} = 0, \text{ gives } \nabla^2 \Phi_{1c} = -B^2 \Phi_{1c} \\ \nabla^2 \Phi_{2c} + B^2 \Phi_{2c} = 0, \text{ gives } \nabla^2 \Phi_{2c} = -B^2 \Phi_{2c} \end{aligned}\right\} \text{ core} \quad (10.21)$$

$$\left.\begin{aligned} \nabla^2 \Phi_{1r} - \kappa_{1r}^2 \Phi_{1r} = 0 \\ \nabla^2 \Phi_{2r} - \kappa_{2r}^2 \Phi_{2r} = 0 \end{aligned}\right\} \text{ reflector} \quad (10.22)$$

The homogeneous equations results for the core allow one to write

$$\begin{aligned} -(D_{1c} B_c^2 + \Sigma_{1c}) \Phi_{1c} + k \Sigma_{2c} \Phi_{2c} = 0 \\ \Sigma_{1c} \Phi_{1c} - (D_{2c} B_c^2 + \Sigma_{2c}) \Phi_{2c} = 0 \end{aligned} \quad (10.23)$$

which have a non-trivial solution if and only if the determinant (Cramer's Rule)

$$\begin{vmatrix} -(D_{1c} B^2 + \Sigma_{1c}) & k\Sigma_{2c} \\ \Sigma_{1c} & -(D_{2c} B^2 + \Sigma_{2c}) \end{vmatrix} = 0 \qquad (10.24)$$

The criticality conditions is, therefore,

$$\begin{aligned} &(D_{1c} B^2 + \Sigma_{1c})(D_{2c} B^2 + \Sigma_{2c}) = k\Sigma_{1c}\,\Sigma_{2c} \\ &k = (1 + L_{1c}^2 B^2)(1 + L_{2c}^2 B^2) \end{aligned} \qquad (10.25)$$

This is a second order equation for (B_m^2) core. The problem is to find the geometrical dimensions which give a geometrical buckling equal to the material buckling. This involves the interplay between the core and the reflector and constitutes the messy part of the calculation.

The two solutions for B^2 are:

$$\begin{aligned} \mu^2 &= \frac{1}{2}\left[-\left(\frac{1}{L_{1c}^2} + \frac{1}{L_{2c}^2}\right) + \sqrt{\left(\frac{1}{L_{1c}^2} + \frac{1}{L_{2c}^2}\right)^2 + \frac{4(k-1)}{L_{1c}^2 L_{2c}^2}} \right] \\ -\nu^2 &= \frac{1}{2}\left[-\left(\frac{1}{L_{1c}^2} + \frac{1}{L_{2c}^2}\right) - \sqrt{\left(\frac{1}{L_{1c}^2} + \frac{1}{L_{2c}^2}\right) + \frac{4(k-1)}{L_{1c}^2 L_{2c}^2}} \right] \end{aligned} \qquad (10.26)$$

The general solution to the core equations are linear combinations of the two solutions determined by μ^2 and $-\nu^2$. Let X by the solution belonging to μ^2 and Y that belonging to $-\nu^2$.

$$\begin{aligned} &\Delta^2 X + \mu^2 X = 0 \\ &\Delta^2 Y - \nu^2 Y = 0 \end{aligned} \qquad (10.27)$$

and

$$\begin{aligned} \Phi_{1c} &= AX + CY \\ \Phi_{2c} &= A' X + C' Y \end{aligned} \qquad (10.28)$$

TABLE 10.1

Reflector

SOLUTIONS OF WAVE EQUATIONS FOR NEUTRON FLUX IN REFLECTOR

Geometry	Z	Z (T infinite)
Infinite slab	$\sinh \kappa_r(1/2H + T - x)$	$e^{-\kappa_r x}$
Sphere	$\frac{\sinh \kappa_r(R + T - r)}{r}$	$\frac{e^{-\kappa_r r}}{r}$
Infinite cylinder	$I_0(\kappa_r r) - \frac{I_0}{K_0}\left[\kappa_r (R + T)\right] K_0(\kappa_r r)$	$K_0(\kappa_r r)$

Core

Geometry	X	Y
Infinite slab	$\cos \mu x$	$\cosh \nu x$
Sphere	$\frac{\sin \mu r}{r}$	$\frac{\sinh \nu r}{r}$
Infinite Cylinder	$J_0(\mu r)$	$I_0(\nu r)$

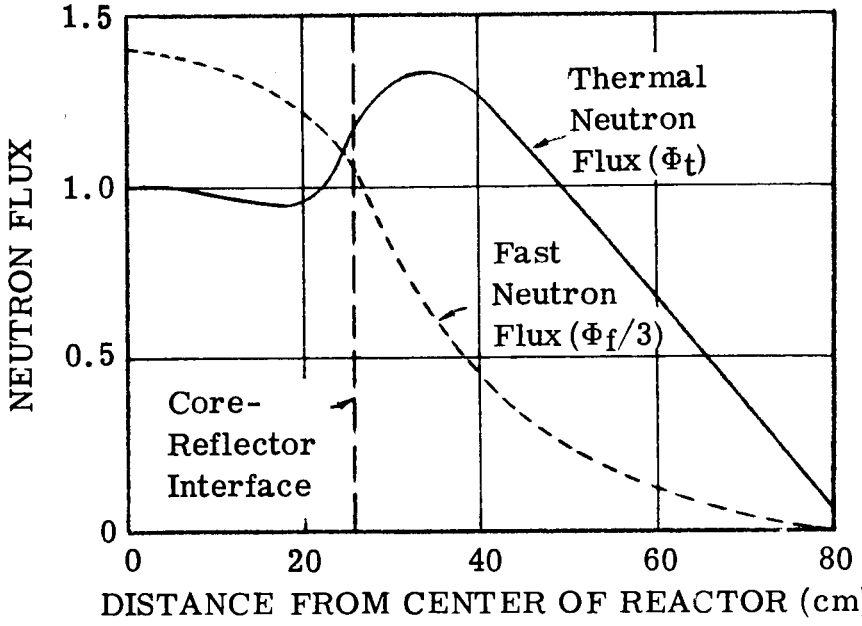


Fig. 10.4 - Typical two-group neutron flux distribution in a reflected slab reactor. The core is an aluminum-water U^{235} mixture for which $U\Sigma_f/\Sigma_{at} = \Sigma_s\xi/\Sigma_{at} = 4.4$; the reflector is beryllium, for which $U\Sigma_f/\Sigma_{at} = \Sigma_{s\xi}/\Sigma_{at} = 97$; and $U = 17.5$ is the lethargy range of the fast group.

Table 10.1 lists the functions X and Y for slab, spherical and cylindrical geometry. It turns out that A and A', and C and C' are proportional to one another so that

$$\Phi_{1c} = AX + CY$$
$$\Phi_{2c} = S_1 AX + S_2 CY \tag{10.29}$$

where

$$S_1 = \left[\frac{D_{1c}}{L_{1c}^2 D_{2c}}\right]\left[\frac{L_{2c}^2}{1+\mu^2 L_{2c}^2}\right]$$
$$S_2 = \left[\frac{D_{1c}}{L_{1c}^2 D_{2c}}\right]\left[\frac{L_{2c}^2}{1-\nu^2 L_{2c}^2}\right] \tag{10.30}$$

Except for A and C, Φ_{1c} and Φ_{1c} and Φ_{2c} are now specified in terms of known numbers.

Similarly the solutions to the reflector equations turn out to be:

$$\Phi_{1r} = F Z_1$$
$$\Phi_{2r} = S_3 FZ_1 + GZ_2 \tag{10.31}$$

where Z_1 and Z_2 are given by Table 10.1 and $S_3 = \frac{\Sigma_{1r}}{D_{2r}}\left[\frac{1}{\kappa_{2r}^2 - \kappa_{1r}^2}\right]$

One determines the constants A, C, F and G by matching the core and reflector fluxes and ~~the first derivatives of these fluxes~~ currents at the core-reflector interface. The matching equation are the following:

$$AX + CY - FZ_1 = 0$$
$$AS_1X + CS_2Y - FS_3Z_1 - GZ_2 = 0$$
$$AD_{1c}X' + CD_{1c}Y' - FD_{1r}Z_1' = 0 \tag{10.32}$$
$$AS_1D_{2c}X' + CS_2D_{2c}Y' - FD_{2r}S_3Z_1' - GD_{2r}Z_2' = 0$$

A non-trivial solution for A, C, F and G exists if and only if the determinant of the coefficients for their unknown vanishes, i.e., if and only if

$$\Delta = \left[D_{1c} \frac{X'}{X} - D_{1r} \frac{Z_1'}{Z_1} \right] \left[S_2 D_{2c} \frac{Y'}{Y} - S_3 D_{2r} \frac{Z_1'}{Z_1} - (S_2 - S_3) D_{2r} \frac{Z_2'}{Z_2} \right]$$

$$- \left[D_{1c} \frac{Y'''}{Y} - D_{1r} \frac{Z_1'}{Z_1} \right] \left[S_1 D_{2c} \frac{X'}{X} - S_3 D_{2r} \frac{Z_1'}{Z_1} - (S_1 - S_3) D_{2r} \frac{Z_2'}{Z_2} \right] = 0$$

One goes about solving this by trial and error. For example, suppose that for a given fuel, moderator, and reflector it is required to find the critical core dimension R for a given reflector thickness T.

1. First use R given by a one-group approximation and evaluate Δ. Δ will in general not be zero but from the value of Δ obtained one can make a good guess about how to change R so Δ will approach zero.
2. Try successive R - guesses. Plot Δ versus R and use the R-value which gives $\Delta = 0$ on the graph.

11. LATTICE THEORY (WW, 610-695)

The resonance escape probability, p, and the thermal utilization, f, vary in opposite directions when the fuel fraction, F, increases, e.g., f increases and p decreases. Because of this, there exists a particular fuel fraction, given a fuel and moderator material, for which the product pf attains a maximum value and, as a consequence, for which the infinite medium multiplication factor

$$k = \eta \epsilon pf \qquad (11.1)$$

also attains a maximum value (approximately). This circumstance suggested that it might be advantagous to collect the reactor fuel into either sheets, rods, spheres, etc. and arrange "fuel elements" periodically within a moderator matrix. Such an arrangement of fuel elements embedded in moderator material is called a reactor lattice. The behavior of a lattice is much different than that of a homogeneous mixture of fuel and moderator. As is well-known, a natural uranium fuel-graphite moderator reactor is workable only when the fuel is segregated in a lattice structure. (See Section 13.) It is impossible to construct a critical homogeneous reactor using graphite moderator and natural uranium fuel.

Localizing the fuel concentration produces the following changes, relative to the characteristics of a homogenized system, in thermal reactors.

1. f is decreased (minor disadvantage)
2. p is markedly increased (major advantage)
3. ϵ is increased (minor advantage)

The double advantage arising from the increase in p and ϵ offsets the relatively small decrease in f by a considerable margin. An outline of the diffusion theory approximation for a reactor lattice will now be given as it pertains to the computation of f and p.

11.1 THERMAL UTILIZATION (THERMAL NEUTRONS)

Let Σ_{ao} and Σ_{a1} be the absorption cross sections, and V_o and V_1 the volumes for fuel and moderator material, respectively. In a homogeneous system

$$f_{hom} = \Sigma_{ao} V_o / (\Sigma_{ao} V_o + \Sigma_{a1} V_1) \qquad (11.2)$$

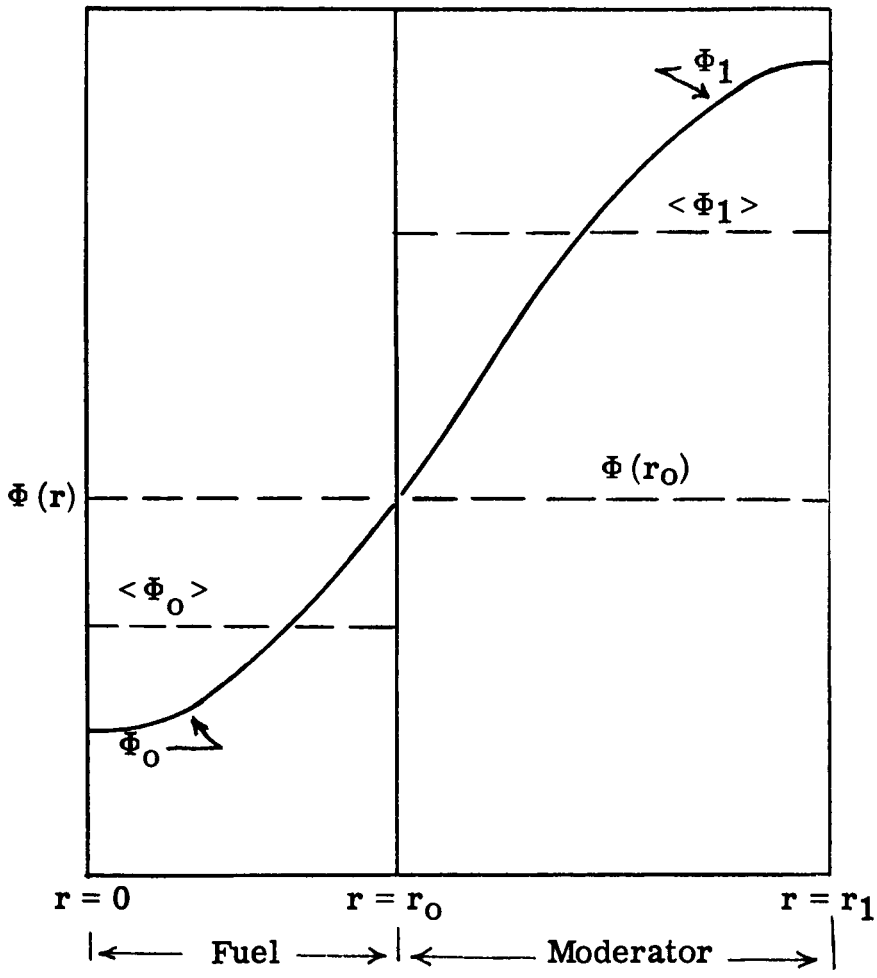

Φ_1
$\langle \Phi_1 \rangle$
$\Phi(r_o)$
$\Phi(r)$
$\langle \Phi_o \rangle$
Φ_o
r = 0
r = r_o
r = r_1
Fuel
Moderator

Figure 11.1

Because the reciprocal form, 1/f, is most convenient in lattice theory, let us write also

$$\frac{1}{f_{hom}} = 1 + \frac{\Sigma_{a1} V_1}{\Sigma_{ao} V_o} \tag{11.3}$$

In a lattice the flux in a fuel element, Φ_o, is generally different from the flux, Φ_1, in the enveloping moderator matrix and the fuel element volume V_o is different from the moderator volume V_1 associated with it in the basic lattice cell.* The thermal utilization for a lattice is

$$f = \frac{\Sigma_{ao} \langle\Phi_o\rangle V_o}{[\Sigma_{ao} \langle\Phi_o\rangle V_o + \Sigma_{a1} \langle\Phi_1\rangle V_1]} \tag{11.4}$$

provided the cross sections Σ_{ao} and Σ_{a1} are independent of position in the fuel and moderator volumes, respectively. The reciprocal form of Eq. (11.4) is

$$\frac{1}{f} = 1 + \frac{\Sigma_{a1} V_1}{\Sigma_{ao} V_o} \left(\frac{\langle\Phi_1\rangle}{\langle\Phi_o\rangle}\right) = 1 + \frac{\Sigma_{a1}}{\Sigma_{ao}} \frac{V_1}{V_o} d \tag{11.5}$$

The name "disadvantage factor" has been ascribed to the ratio

$$d = \langle\Phi_1\rangle / \langle\Phi_o\rangle \geq 1 \tag{11.6}$$

Assume a cell with fuel contained in the region $o<r<r_o$ and moderator in the region $r_o<r<r_1$, $r=r_1$, being the exterior boundary of the cell. Figure 11.1 describes the general behavior of the thermal flux in a lattice cell. Because $\langle\Phi_1\rangle / \langle\Phi_o\rangle$ is greater than unity one sees that

$$\frac{1}{f} - \frac{1}{f_{hom}} > 0$$

so that $f < f_{hom}$ whenever the flux behaves as it does in Figure 11.1. We will now show that this is, indeed, always the case.

According to diffusion theory the neutron balance equations for the cell are

*The basic lattice cell is a module of fuel and moderator of such geometry that a repeated "pattern" of this module gives the reactor lattice.

$$D_0 \Delta\Phi_0 - \Sigma_{a0} \Phi_0 = 0 \qquad 0<r<r_0 \text{ (fuel)} \qquad (11.7)$$

$$D_1 \Delta\Phi_1 - \Sigma_{a1} \Phi_1 + q = 0 \qquad r_0<r<r_1 \text{ (moderator)} \qquad (11.8)$$

where q is the thermal neutron source term (neutrons per cm^3 per sec). There are three boundary conditions for these equations. Two of them are the usual conditions on the continuity of the flux and current at a material interface - which is $r=r_0$ in this instance.

$$\Phi_0(r_0) = \Phi_1(r_0) \qquad \text{Bnd Condition (1)} \qquad (11.9)$$

$$D_0 \nabla_n \Phi_0(r_0) = D_1 \nabla_n \Phi_1(r_0) \qquad \text{Bnd Condition (2)} \qquad (11.10)$$

The symbol ∇_n means "directional derivative along the normal to the surface in queation." The third boundary condition

$$\nabla_n \Phi_1(r_1) = 0 \qquad \text{Bnd Condition (3)} \qquad (11.11)$$

stipulates that the net thermal neutron flow between adjoining cells is zero. Our balance equations under these conditions have the solutions

$$\Phi_0(r) = \alpha R(\kappa_0 r) \qquad (11.12)$$

$$\Phi_1(r) = q/\Sigma_{a1} - \beta C(r) \qquad (11.13)$$

$$\text{where } \alpha = - D_1 C'(r_0)/D_0 \kappa_0 R'(\kappa_0 r_0) \qquad (11.14a)$$

$$q/\Sigma_{a1} = [C(r_0) - D_1 C'(r_0) R(\kappa_0 r_0)/D_0 \kappa_0 R'(\kappa_0 r_0)] \qquad (11.14b)$$

and β can be taken as unity. The functions R, R', C and C' are defined in Table 11.1 for three cell geometries.

One uses the solutions for Φ_0 and Φ_1 to obtain f in the following way:

1. Observe that the production rate of thermal neutrons is $q V_1$ if one assumes the thermal neutron production in the fuel is negligible.

2. Observe that in an infinite system (the lattice not the cell) the number of neutrons absorbed is equal to the number produced. Since all cells are identical then $q V_1$ is identical for all cells and hence the number of neutrons absorbed in each cell must be $q V_1$ since Eq. (11.11) states there is no net neutron flow from one cell to another.
3. Observe that the number of neutrons absorbed in the fuel element must be equal to the total neutron flow into the fuel element. This total flow is

   Net flow into fuel element = $D_1 \nabla \Phi_1 (r_0) \cdot S$

   where S is the fuel element surface area and $D_1 \nabla \Phi_1 (r_0)$ is the net current into the fuel element at its surface $r=r_0$.

This physical reasoning gives.

$$f = \frac{\text{Number of Neutrons Absorbed in Fuel element}}{\text{Number of Neutrons Absorbed in Cell}} \qquad (11.15)$$

$$= \frac{S D_1 \; n \Phi_1(r_0)}{q V_1} = - \frac{S D_1 C'(r_0)}{q V_1}$$

$$\frac{1}{f} = \frac{q V_1}{-S D_1 C'(r_0)} \qquad (11.16)$$

($C'(r)$ is always negative so $1/f$ is positive)

Substitution Eq. (11.14b) into Eq. (11.16) leads to

$$\frac{1}{f} = \frac{C(r_0) \Sigma_{a1} V_1}{-S D_1 C'(r_0)} + \frac{R(\kappa_0 r_0) \Sigma_{a1} V_1}{R'(\kappa_0 r_0) S D_0 \kappa_0} \qquad (11.17)$$

By convention Eq. (11.17) is written also as

$$\frac{1}{f} = 1 + X + R_a \qquad (11.18)$$

where $1+X = \dfrac{C(r_0) \Sigma_{a1} V_1}{-S D_1 C'(r_0)}$ (excess absorption) (11.19)

$$R_a = \frac{R(\kappa_0 r_0) \Sigma_{a1} V_1}{R'(\kappa_0 r_0) S D_0 \kappa_0} \text{ (relative absorption)} \qquad (11.20)$$

In this representation the words "excess" and "relative" refer to what would be the case if the average moderator flux $\langle\Phi_1\rangle$ had the value $\Phi(r_o)$, i.e., the interface value.

$$R_a = \frac{\text{Moderator Absorption (If } \langle\Phi_1\rangle \text{were } \Phi(r_o))}{\text{Fuel absorption}} \tag{11.21}$$

$$1+X = 1 + \frac{\text{Moderator Absorption} - \Sigma_{a1} V_1 \ \Phi(r_o)}{\text{Fuel Absorption}} \tag{11.22}$$

It is now possible to show that $f < f_{hom}$ because

$$\frac{1}{f} - \frac{1}{f_{hom}} = 1 + X + R - \left(1 + \frac{\Sigma_{a1} V_1}{\Sigma_{ao} V_o}\right) \tag{11.23}$$

$$= X + \frac{\Sigma_{a1} V_1}{\Sigma_{ao} V_o} \left[\frac{\Phi(r_o)}{\langle\Phi_o\rangle} - 1\right] > 0$$

This difference is positive because $\langle\Phi_o\rangle$ is less than $\Phi(r_o)$.

TABLE 11.1

THE FUNCTIONS R, R', C AND C' ARE LISTED BELOW FOR THREE CELL GEOMETRIES

Geometry	$R(\kappa_i r)$	$R'(\kappa_i r)$	$I(\kappa_i r)$	$I'(\kappa_i r)$
Plane	$\cosh \kappa_i r$	$\sinh \kappa_i r$	$\sinh \kappa_i r$	$\cosh \kappa_i r$
Cylinder*	$I_0(\kappa_i r)$	$I_1(\kappa_i r)$	$-K_0(\kappa_i r)$	$K_1(\kappa_i r)$
Sphere	$\frac{\sinh \kappa_i r}{\kappa_i r}$	$\frac{\kappa_i r \cosh \kappa_i r - \sinh \kappa_i r}{(\kappa_i r)^2}$	$-\frac{\cosh \kappa_i r}{\kappa_i r}$	$\frac{-\kappa_i r \sinh \kappa_i r + \cosh \kappa_i r}{(\kappa_i r)^2}$

$$C(r) = I'(\kappa_1 r_1)\, R(\kappa_1 r) - R'(\kappa_1 r_1)\, I(\kappa_1 r)$$

$$C'(r) = \kappa_1 \left[I'(\kappa_1 r_1)\, R'(\kappa_1 r) - R'(\kappa_1 r_1)\, I'(\kappa_1 r) \right]$$

*The Bessel functions I and K used here are those defined in Watson's Bessel Functions and are always positive.

11.2 RESONANCE ESCAPE PROBABILITY

Define a "resonance" neutron utilization factor f_R in exactly the same way the thermal (neutron) utilization factor is defined.

$$f_R = \frac{\text{Number of resonance neutrons absorbed in the fuel}}{\text{Total number of resonance neutrons absorbed}} \qquad (11.24)$$

Proceeding formally in exactly the same way as in Section (11.1) one finds

$$\frac{1}{f_R} = 1 + X_{(R)} + R_{a(R)} \qquad (11.25)$$

p can be shown to have the form

$$p = e^{-f_R/(1-f_R)} \qquad (11.26)$$

Now because f_R will be smaller than $f_R(\text{hom})$, for the same reasons $f < f_{hom}$, it follows that

$$p > p_{hom} \qquad (11.27)$$

This follows because given x and $\epsilon > 0$

$$e^{-\frac{x}{1-x}} > e^{\frac{-(x+\epsilon)}{[1-(x+\epsilon)]}} \qquad (11.28)$$

where the association $f_R = x$ and $f_R(\text{hom}) = x+\epsilon > f_R$ has been made to simplify notation. Inequality (11.28) is true because the exponent on the left

$$\frac{x}{1-x}$$

is smaller than the exponent on the right

$$\frac{x+\epsilon}{1-(x+\epsilon)}$$

by the amount ϵ.

It has now been shown that, diffusion theory, predicts that lumping the fuel into a fuel element (1) decreases f and (2) increases p relative to their values were the fuel homogeneously distributed throughout the moderator.

11.3 DISADVANTAGE FACTOR

From Equations (11.5) and (11.17) the disadvantage factor is

$$d = \frac{\Sigma_{ao} V_o}{\Sigma_{a1} V_1} \left(\frac{1}{f} - 1 \right) = \frac{\Phi(r_o)}{\langle \Phi_o \rangle} + \frac{\Sigma_{ao} V_o}{\Sigma_{a1} V_1} X \qquad (11.29)$$

The disadvantage factor measures the loss in thermal utilization, suffered by lumping the fuel, as a consequence of the thermal flux depression which occurs in the fuel element relative to the surface flux $\Phi(r_o)$. $d > 1$ because thermal neutrons are produced, by slowing down of fast neutrons, in the moderator. These thermal neutrons then diffuse toward the fuel elements which are strong sinks (absorbers) for thermal neutrons and therefore cause a flux depression within themselves; a general characteristic of any strong sink which is fed by an external source.

If the moderator diffusion coefficient were infinite the second term in Equation (11.29) would vanish and d would arise purely from the fuel element self-shielding term $\Phi(r_o)/\langle \Phi_o \rangle$. In the Oak Ridge X-10 reactor the self shielding part of d is 1.1 and the total disadvantage factor is 1.8. Because the minimum value for d is unity, one sees that, in this reactor, non-uniformity of the flux in the moderator is about 7 times more important a contributor to d than is flux non uniformity in the fuel.

For a given ratio V_o/V_1 and equal fuel element volume to surface ratios, f is largest for slab fuel elements and least for spherical fuel elements. The important difference between these geometries being that in the slab geometry there is no shrinking in the areal cross section as the diffusing neutrons approach the fuel element. This shrinking of areal cross section for fuel elements with curved surfaces is called "bottle necking." Bottle necking always decreases f. One way to partially cure bottle necking is to place an "air" gap between the fuel element surface and the surrounding moderator.

12. EFFECTIVE RESONANCE INTEGRAL - ELEMENTARY THEORY (GE, 252-257)

Before a detailed knowledge of the energy behavior of neutron cross sections existed and before the advent of high-speed computers, the resonance escape probability was determined experimentally. It was then expressed in terms of a quantity called the effective resonance integral. In approximate form

$$p(u_o \rightarrow u) = e^{\left[-\int_{u_o}^{u} \frac{\Sigma_a}{\xi\Sigma}\, du\right]} \tag{12.1}$$

If one assumes that $\Sigma_a \cong \Sigma_{ao}$ for the fuel, then it follows that

$$\frac{\Sigma_a}{\Sigma} = \frac{N_o}{\Sigma_s}\left(\frac{\sigma_{ao}\Sigma_s}{\Sigma}\right) \tag{12.1}$$

where N_o is the number of fuel nuclei per cm^3. The effective microscopic absorption cross section $(\sigma_{ao})_{eff}$ is defined as

$$(\sigma_{ao})_{eff} \equiv \frac{\sigma_{ao}\Sigma_s}{\Sigma} = \frac{\sigma_{ao}}{1 + \frac{N_o \sigma_{ao}}{\Sigma_s}} \tag{12.3}$$

The quotient Σ_s/N_o is thought of as the scattering cross section per fuel nucleus, i.e., per absorber nucleus, and the effective resonance integral, I_{eff}, is defined as

$$I_{eff} = \int_{u_o}^{u} (\sigma_{ao})_{eff}\, du \tag{12.4}$$

In terms of I_{eff},

$$p = e^{\left[-\frac{N_o I_{eff}}{\xi \Sigma_s}\right]} \tag{12.5}$$

It must be assumed that Σ_s is constant in lethargy in this representation of p.

In pure fuel $N_o = N$ (total number of nuclei per cm^3 and one has

$$(\sigma_{ao})_{eff} = \frac{\sigma_{ao}}{1 + \frac{\sigma_{ao}}{\sigma_s}} < \sigma_{ao} \quad N_o \to N$$

If the fuel constant is very dilute, i.e., $N_o \to 0$, one has

$$(\sigma_{ao})_{eff} = \sigma_{ao} \qquad N_o \to 0$$

Hence I_{eff} takes its minimum value in pure fuel and its maximum value in a dilute fuel system.

The pre-factor $(N_o/\xi\Sigma_s)$ in the exponent of Equation (12.5) behaves in an opposite manner. Its maximum value occurs for $N_o = N$, its minimum value for $N_o = 0$. The value of p, therefore, depends on two oppositely varying factors. The "advantage" in the tendency for p to decrease with increasing N_o, associated with the decrease in $(\sigma_{ao})_{eff}$, is measured by the so-called volume advantage factor, VAF,

$$VAF = \frac{\sigma_{ao}}{(\sigma_{ao})_{eff}} = 1 + \frac{\sigma_{ao}}{\sigma_{so} + \frac{N1}{N_o}\sigma_{s1}} \tag{12.6}$$

where N_1 is the number of moderator nuclei per cm^3. Physically, VAF is the reciprocal of the scattering probability. (Prove this statement.)

From elementary slowing down theory (GE, 255; 147-160) the neutron flux as a function of energy behaves as

$$\Phi(E) \propto \frac{1}{E} \frac{1}{(\sigma_{ao} + \text{constant})} \tag{12.7}$$

Hence if there is a large amount of fuel the resonance flux is depressed. (Resonance flux is the neutron flux for the neutron energy range in which the exceptionally large resonance absorption cross section peaks occur for U^{238} or U^{235}). This explains why the effective resonance integral decreases as N_o increases. Figure 12.1 illustrates this behavior. Because of the depression in resonance flux, the product $\Sigma_a\Phi(E)$ is smaller in the resonance region than it would be if $\Phi(E)$ were not depressed and the number of resonance absorptions is decreased. The U^{238} cross section in the resonance region is plotted in Figure 12.2.

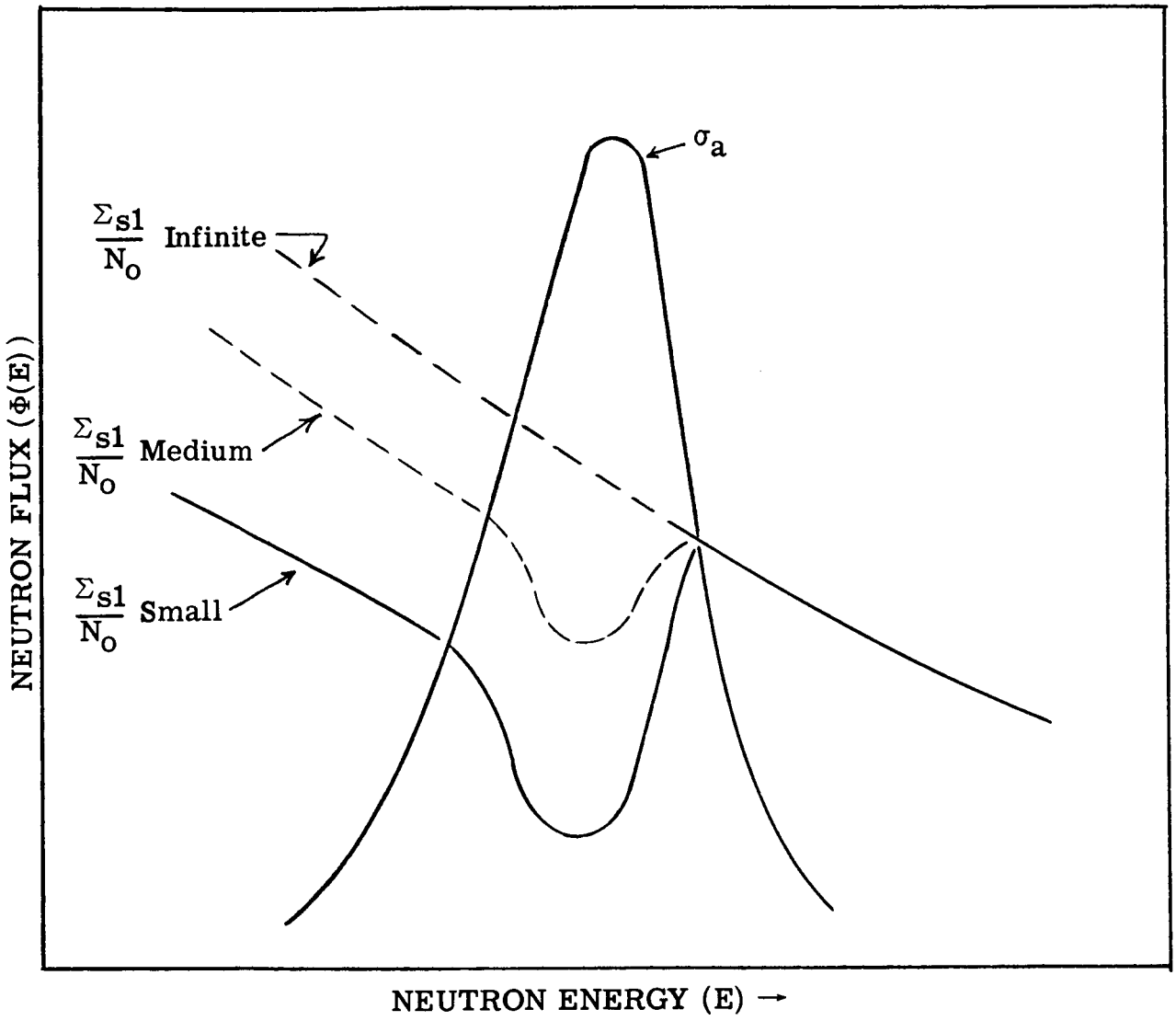


Fig. 12.1 - Resonance flux depression

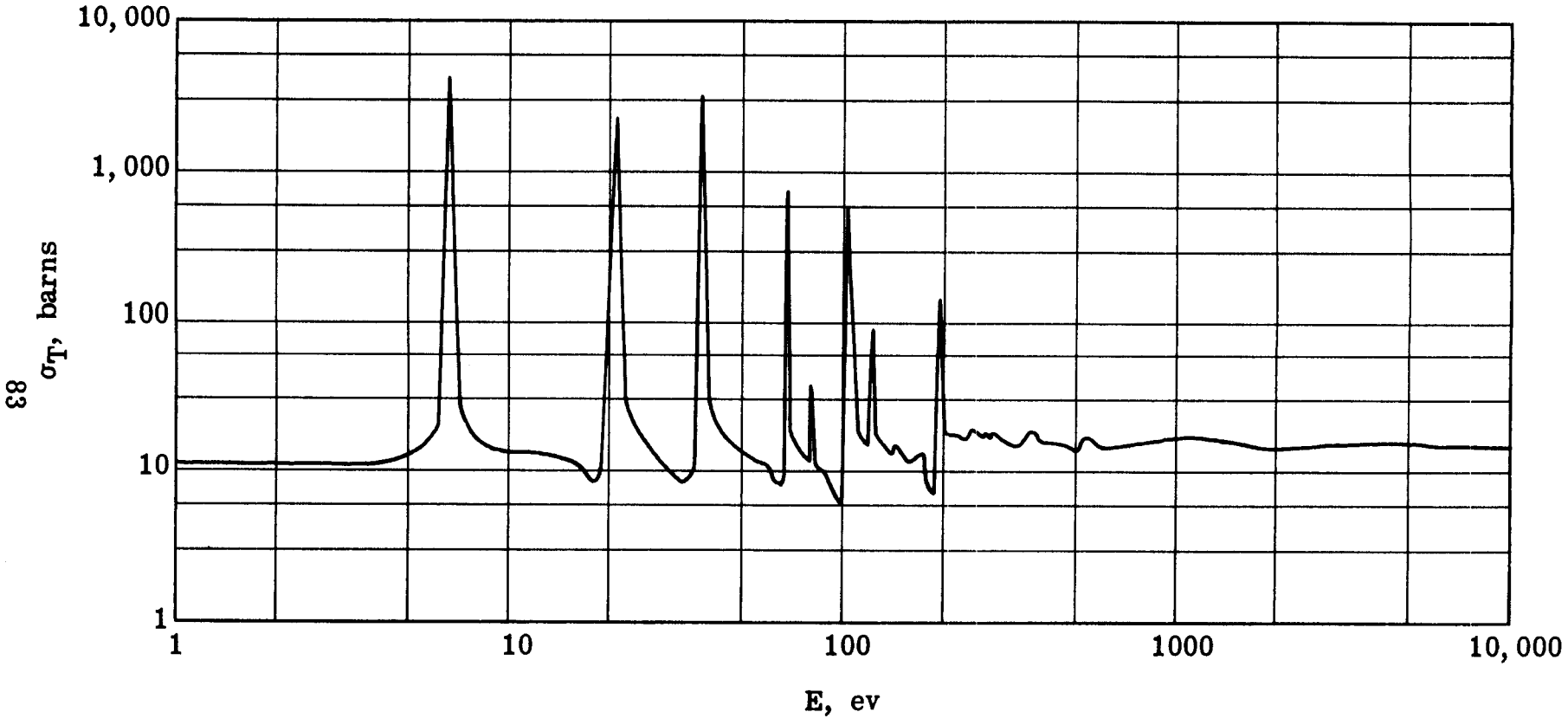


Fig. 12.2 - Total cross section of U^{238} in resonance region

13. EFFECTIVE RESONANCE INTEGRAL - REACTOR LATTICE

I_{eff}, in barns, is defined such that

$$V_o N_o I_{eff} \tag{13.1}$$

is the total number of absorptions in a fuel element of volume V_o with N_o fuel nuclei per unit volume for a unit flux per unit lethargy. I_{eff} is determined experimentally and expressed in terms of two types of empirical equations

$$I_{eff} = I_M + I_s \left(\frac{S}{M}\right)^{1/2} \tag{13.2a}$$

$$I_{eff} = \frac{<\Phi_o>}{\Phi_o(r_o)} I_M + I_s \left(\frac{S}{M}\right) \tag{13.2b}$$

the latter being the older form. Usually it is measured with respect to the flux at the fuel element surface $\Phi(r_o)$ and in this case flux depression in the fuel element is automatically accounted for in the experimental measurement.

In other instances it is measured such that the mass term I_m must be weighted by $<\Phi_o>/\Phi(r_o)$ to account for flux depression in the fuel.

13.1 I_{eff} DEFINED FROM EQUATION (13.2a)

By definition the fuel region absorption is

$$V_o N_o I_{eff} \frac{\Phi(r_o)}{\Delta U_R} = D_1 S \left.\frac{\partial \Phi_1}{\partial r}\right|_{r=r_o} \tag{13.3}$$

In this case

$$\frac{1}{f_R} = \frac{q V_1}{D_1 S \left.\frac{\partial \Phi}{\partial r}\right|_{r=r_o}} = \frac{\Sigma_{a1} V_1 C(r_o)}{-S D_1 C'(r_o)} + \frac{\Sigma_{a1} V_1 \Delta U_R}{V_o N_o I_{eff}} \tag{13.4}$$

as

$$\begin{aligned} q &= [\beta C(r_o) + \alpha R(\kappa_o r_o)] \Sigma_{a1} \\ &= [\beta \Phi_1(r_o) + \Phi_o(r_o)] \Sigma_{a1} \end{aligned} \tag{13.5}$$

$$\frac{1}{f_R} = \frac{q V_1}{D_1 S \left(\frac{\partial \Phi 1}{\partial r}\right)_{r=r_o}} = \frac{\beta \Sigma_{a1} V_1 C(r_o)}{-S \beta D_1 C'(r_o)} + \frac{\Sigma_{a1} V_1 \Phi_o(r_o) \Delta U_R}{V_o N_o I_{eff} \Phi_o(r_o)}$$

$$= \frac{\Sigma_{a1} V_1 C(r_o)}{-S D_1 C'(r_o)} + \frac{\Sigma_{a1} V_1 \Delta U_R}{V_o N_o I_{eff}} \qquad (13.6)$$

If one uses $\Sigma_{a1} = \frac{\xi \Sigma_1}{\Delta U_R}$ it follows that

Basic Equation

$$\frac{1}{f_R} = \frac{\xi \Sigma_1 V_1 C(r_o)}{-S D_1 \Delta U_R C'(r_o)} + \frac{\xi \Sigma_1 V_1}{N_o V_o I_{eff}} \qquad (13.7)$$

13.2 I_{eff} DEFINED FROM EQUATION (13.2b)

If I_{eff} is taken as

$$I_{eff} = I_M \frac{\langle \Phi_o \rangle}{\Phi_o(r_o)} + I_s \frac{S}{M} \qquad (13.8)$$

then from Equation (13.7)

$$\frac{1}{f_R} = \frac{\xi \Sigma_1 V_1 C(r_o)}{-S D_1 \Delta U_R C'(r_o)} + \frac{\xi \Sigma_1 V_1}{N_o V_o \left\{ I_M \frac{\langle \Phi_o \rangle}{\Phi_o(r_o)} + I_s \left(\frac{S}{M}\right) \right\}} \qquad (13.9)$$

But

$$\frac{\langle \Phi_o \rangle V_o}{\Phi_o(r_o)} = \kappa_o^{-1} \alpha S R'(\kappa_o r_o) \qquad (13.10)$$

So

$$N_o V_o I_M \frac{\langle \Phi_o \rangle}{\Phi_o(r_o)} = \frac{N_o I_M}{\alpha R(\kappa_o r_o)} \kappa_o^{-1} \alpha S R'(\kappa_o r_o)$$

$$= \frac{N_o I_M S R'(\kappa_o r_o)}{\kappa_o R(\kappa_o r_o)} \qquad (13.11)$$

$$\frac{1}{f_R} = \frac{\xi \Sigma_1 V_1 C(r_o)}{-S D_1 \Delta U_R C'(r_o)} + \frac{\xi \Sigma_1 V_1}{V_o N_o \left\{ \frac{I_M S R'(\kappa_o r_o)}{V_o \kappa_o R(\kappa_o r_o)} + I_s \left(\frac{S}{M}\right) \right\}} \qquad (13.12)$$

This is a good representation for f_R

In this equation:

$$D_o = 1/3\Sigma$$

$$\Sigma_{ao} = \frac{N_o I_{eff}}{\Delta U_R} \qquad (13.13)$$

$$\kappa_o = \sqrt{\frac{N_o I_{eff}}{\Delta U_R D_o}}$$

(ΔU_R is the width of the resonance region in lethargy units)

13.3 EMPIRICAL EXPRESSION FOR I_{eff}

$$I_{eff} = \left[2.8 + 25 \left(\frac{S}{M}\right)^{1/2} \right] \times 10^{-24}\ cm^2$$

$$I_{eff} = \left[8.0 + 27.5 \left(\frac{S}{M}\right) \right] \times 10^{-24}\ cm^2 \quad \text{Uranium metal}$$

$$I_{eff} = \left[4.15 + 26.6 \left(\frac{S}{M}\right)^{1/2} \right] \times 10^{-24}\ cm^2$$

$$I_{eff} = \left[11.0 + 24.5 \left(\frac{S}{M}\right) \right] \times 10^{-24}\ cm^2 \quad UO_2$$

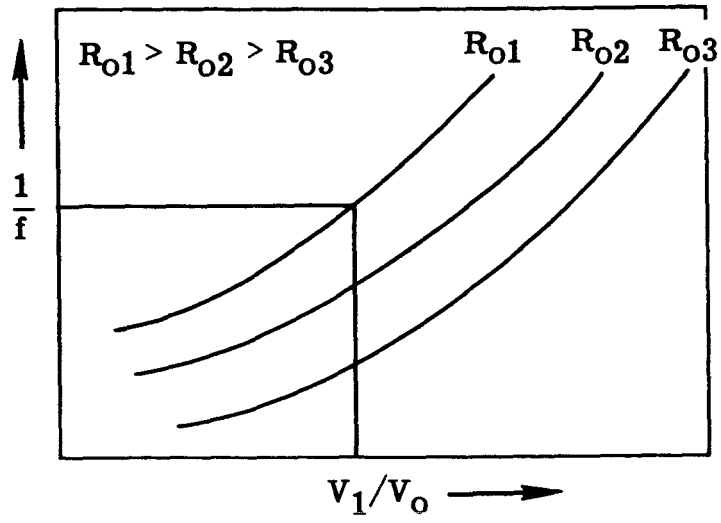
I_M I_s

URANIUM METAL, I_{eff} IN BARNS

S/M	0.0312	0.0625	0.125	0.250
I_{eff} (Ex)	7.21	8.72	11.26	15.21
I_{eff} (Theory)	7.1	8.9	11.6	15.3

Value $\cong$ 240 with ∞ dilution

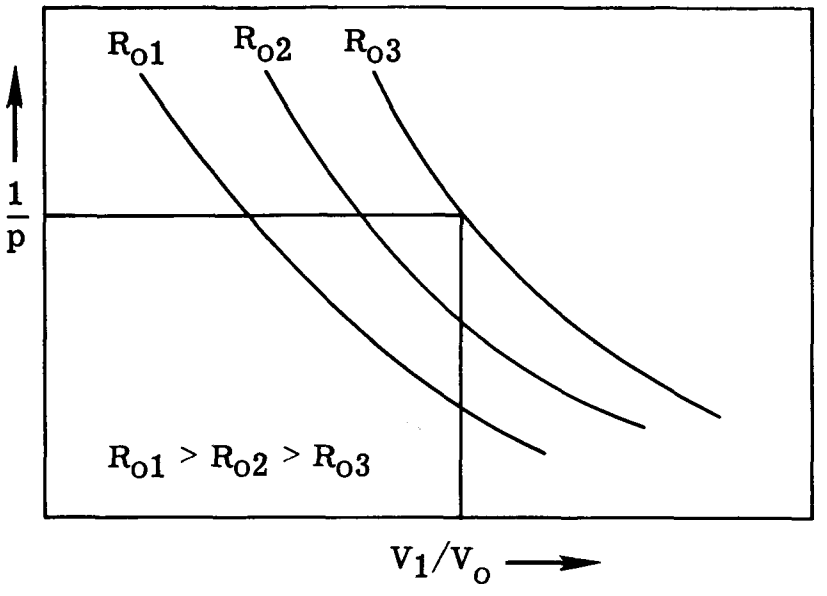
13.4 $\frac{1}{f}$ AND $\frac{1}{p}$ COMPARED



a. $\frac{1}{f}$ increases with increasing R_o at constant V_1/V_o

i. e., f decreases with increasing R_o at constant V_1/V_o because of flux depression in fuel

b. $\frac{1}{f}$ increases with increasing V_1/V_o at constant R_o because of more absorption in moderator



a. $\frac{1}{p}$ decreases with increasing R_o at constant V_1/V_o

i. e., p increases (flux depression)

b. $\frac{1}{p}$ decreases with increasing V_1/V_o at constant R_o

i. e., p increases (more activity in moderator)

Optimum values for p and f therefore exist at which the product p f is maximum. In this respect it is interesting to observe how p f behaves in a homogeneous natural uranium-graphite system. The values of I_{eff}, p, f and p f are listed in Table 13.1 for this system.

TABLE 13.1

ABSORPTION PARAMETERS FOR HOMOGENEOUS NATURAL URANIUM GRAPHITE SYSTEM

Ni/No	I_{eff}	p	F	pF
200	72 barns	0.579	0.889	0.515
300	87	0.643	0.842	0.541
400	100	0.682	0.800	0.546
500	112	0.693	0.762	0.528

This system cannot be made critical because $(pf)_{max} \cong 0.55$ is too small to give

$$k = \eta \epsilon p f = 1$$

Because $\eta = 1.3$ and $\epsilon = 1.03$ for natural uranium fuel, $\eta\epsilon = 1.34$ and hence pf must exceed $1/1.34 = 0.745$ for the system to be critical as an infinite system. The fact that $(pf)_{max} = 0.55 < 0.745$ precludes criticality.

An ingenious solution to this problem was devised by Wigner and Szilard which was applied to the construction of the first nuclear reactor. This was to lump the uranium metal in balls, separated by graphite. Since the uranium atoms do not slow neutrons down significantly, the neutrons which are produced at energies higher than the resonance energies in the uranium will diffuse into the graphite region. These, then stand a good chance of slowing down past the resonance region before colliding with another uranium atom. Obviously the distance between lumps is a very crucial parameter of the heterogeneous reactor. As the lumps become further apart, we saw that f will decrease and p will increase. There is therefore again an optimum lump size and spacing for maximum pf.

In a finite reactor the leakage of neutrons must also be accounted for.

14. REACTOR CONTROL AND FISSION PRODUCT POISONING

Obviously, more fuel than necessary for criticality must be loaded into a reactor which is to operate at a measurable power level. In order to get up to a significant power, the reactor must be made supercritical for a time. Also, as fissions occur, tending to reduce the multiplication factor there must be some method of raising the multiplication factor continuously.

The rate of fuel burnup can be estimated from the energy release per fission. 3.1×10^{10} fissions per second are required to produce 1 watt of power. The rate of fuel burnup will be approximately one gram per megawatt-day. For low power reactors thic can be almost negligible as far as determining excess reactivity requirements.

14.1 FISSION PRODUCT POISONING

An important factor in the excess reactivity requirement is the fission product buildup. When the reactor is designed, materials of as low an absorption cross section as is practical are used. However, as fission begins, the fission fragments begin to contaminate the core. As far as the effect of the fission products on the multiplication factor goes, they can be divided into two categories - short lived and long lived isotopes. The long lived isotopes buildup steadily in a manner proportional to the power-time history of the reactor. In Figure 14.1 obtained by R. N. Deutsch and published in a memo at G. E.'s Knolls Atomic Power Laboratory, the effect of stable fission product poisoning is plotted in terms of equivalent grams of B^{10} per kilogram of fuel as a function of fractional fuel burnup. To perform a "burnout" calculation, one would first compute the amount of fuel destroyed in a given time from the known power level of the reactor and then use Figure 14.1 to derive the fission-product poisoning effect in terms of equivalent B^{10}. Then the k of the reactor can be computed using a B^{10} cross section of 3810 barns. If the reactor was initially critical ($k=1$) a value of $k<1$ will be derived for the burnt-out reactor. The eifference in k must be compensated for by control.

A more important form of the poisons is the short-lived, high cross section poison Xe^{135}. This isotope has the highest known absorption cross section of any isotope (about 3.5×10^6 barns). It has a half-life of about 14 hours. Although Xe^{135} is formed as a fission product to a small extent, the major source of Xe^{135} is by the radioactive decay of I^{135} which is the daughter of Te^{135}, a fission product.

The concentration of Xe^{135} as a function of time is determined by the rate of buildup, due to flux and decay of Iodine, and the rate of loss which

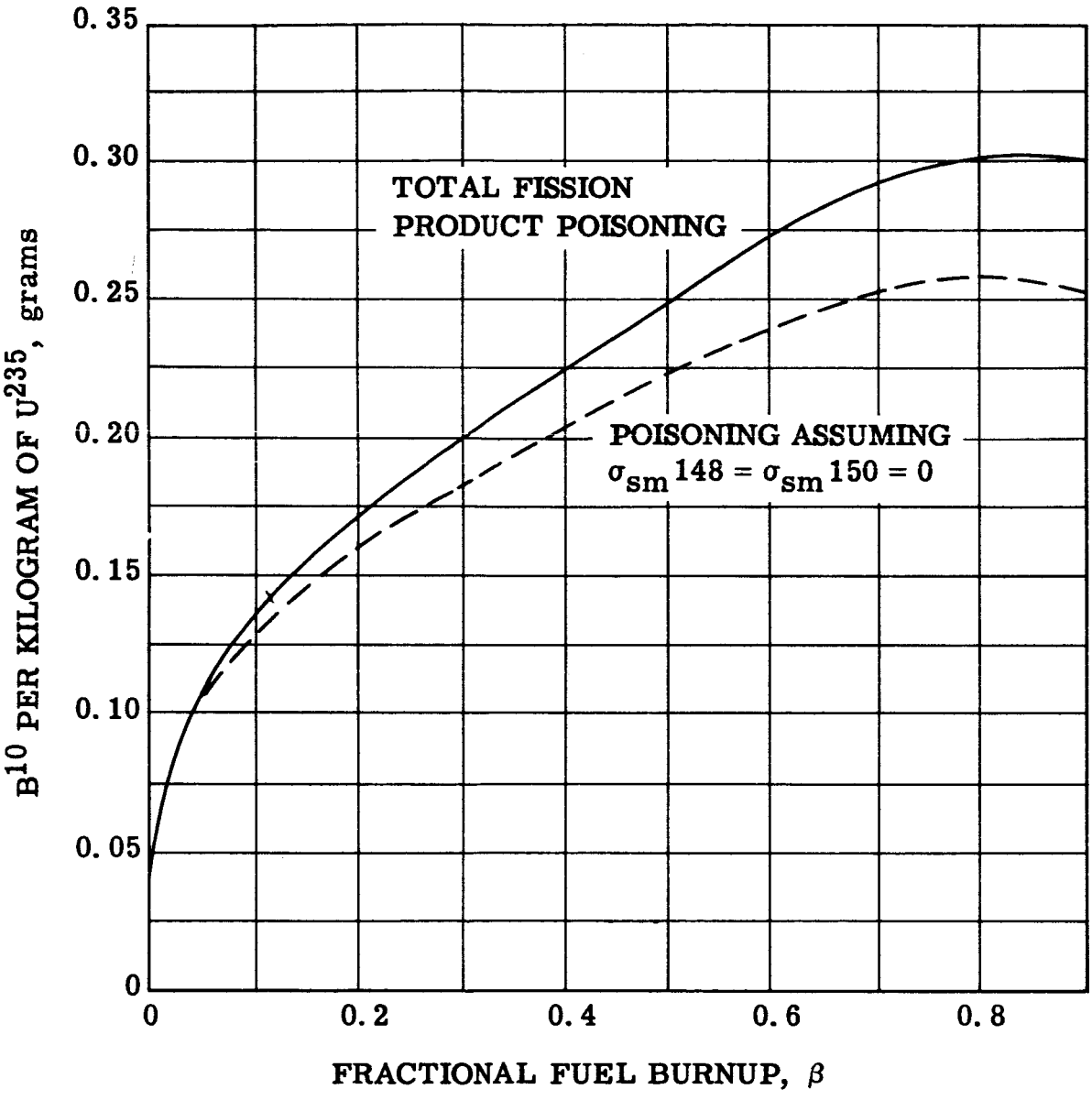


Fig. 14.1 - Fission product buildup in enriched thermal reactors

is due to Xe^{135} burnup (neutron absorption) and Xe^{135} decay. These two competing effects result in an equilibrium concentration occurring after approximately 50 hours of steady operation. For a high flux reactor ($\Phi \approx 10^{14}$ neuts/cm^2-sec) the equilibrium effect on reactivity is about 5 percent. This in itself is not excessive. However, if after this concentration is built up, the reactor is shut down the Xenon concentration will begin to build up rapidly. This is due to the fact that the neutron burnup term disappears. Xenon is still being formed, however at the same rate, due to the decay of Iodine. For a high flux reactor, this effect will peak at about 12 hours after shutdown with a reactivity effect of almost 40 percent, decaying after about two days to negligible proportions. If it is necessary to restart during the period of large buildup, a large amount of extra fuel (xenon override) must be present in the reactor. It should be mentioned that since the fission product poisons are due to large thermal absorption cross sections, reactors in which most of the fissions are produced by non-thermal neutrons, namely intermediate or fast reactors, do not exhibit this poisoning effect.

14.2 REACTOR CONTROL

Another effect requiring excess reactivity is due to high temperature operation. To be stable, reactors are designed to have negative temperature coefficients of reactivity. Then, if something causes the power to rise, the temperature of the reactor will rise and the reactivity will become less than 1, reducing the power to its original level. This of course means that when the reactor is started up from the cold condition, reactivity must be added as the average temperature of the reactor increases.

In order to compensate for the extra reactivity until it is needed, two methods are available.

Control rods, the most common method employed to date consist of rods filled with a poison such as boron or cadmium. The rods can be inserted into the reactor to reduce reactivity and removed to increase it. (See Appendix for optional location.) The insertion of a poison rod can also be synchronized with the removal of a fuel rod to increase the effect. This method, though effective, leads to local distortions of flux which tend to reduce the efficiency of the power production.

A more recent solution to the reactor control problem is the use of burnable poisons. This is the insertion into the reactor, in a permanent fashion, neutron poisons which, partially at least, compensate for the

excess fuel which burns out with time. In theory, the poison should have the identical cross section as the fuel, so that as fuel burns out, the poison burns out at the same rate, the reactivity remaining constant with time. At present boron, having a cross section the closest to the desired value, is used, but tends to burn out faster than the fuel. Thus the excess reactivity begins to increase with initial operation of the reactor, and then decreases.

The discussion on controls in this section has implied solid fueled reactors. In the case of liquid fuel reactors, which have not been exploited successfully as yet, the problems of control are largely alleviated. Fuel solutions can be diluted or strengthened continuously as required. Also xenon is an insoluble gas in these reactors and this problem disappears. This situation is one of the major advantages of the liquid fueled (either aqueous or liquid metal solution, or slurry) reactors.

15. REACTOR DESIGN PROBLEM

Part One: Diffusion Kernel Computation

Design a minimum volume bare cylindrical reactor to operate at 500^{o}F. This reactor is to furnish 30 megawatts of power for 30 days. (No less than 30 days and no more than 35 days.) Assume 3.1×10^{16} fissions/sec will produce 1 megawatt of power. Allow no more than 0.15 excess criticality factor at the beginning of operation. Summarize the characteristics of the reactor by giving:

1. Physical dimensions, radial buckling and longitudinal buckling.
2. Fuel loading.
3. Excess criticality at the start of operation.
4. Plot of longitudinal and radial thermal flux.
5. Plot of excess criticality as a function of operating time.
6. Plot of control system capacity required to maintain critical operation as a function of operating time.
7. Fast neutron non-escape probability.
8. Thermal neutron non-escape probability.

The computational work is to be based on the following specifications:

1. Use a water moderator (ordinary water) and U-235 fuel. These are to be uniformly mixed and the finite volume of the fuel material accounted for in the computation.
2. Treat fast neutron slowing down according to the four kernel (diffusion kernel) method summarized in Section 9.3.
3. Compute power from the equation

$$P(\text{megawatts}) = (3.1 \times 10^{16})^{-1} \int \int \epsilon \, \Sigma_f \Phi_t(r, z) \, dr \, dz$$

4. Use uniformly distributed cadmium as a control system. Assume the microscopic absorption cross-section for thermal neutrons is 3700 barns but is zero for all other energies.
5. Assume that poisoning from fission products can be represented by the method of using an equivalent amount of Boron-10 absorption with the Boron-10 uniformly distributed. Assume the microscopic absorption cross section for thermal neutrons is 4000 barns but is zero for all other energies. The amount of Boron-10 to use in this method is given in Table 1.0 as a function of the amount of U-235 consumed during operation.

TABLE 1.0

Fractional Burn-up	Grams of Boron-10 per Kg of U-235
0	0
0.025	0.085
0.050	0.105
0.075	0.120
0.100	0.135
0.150	0.155
0.200	0.170
0.300	0.200
0.400	0.225
0.500	0.250

6. All cross section information required is given in Table 2.0.

Part Two: Multigroup Method

Set up the multigroup equations for a Be-U-235 spherical reactor. Evaluate the group constants. This will not be a thermal reactor and in the multigroup method fast fissions are explicity accounted for in each fast group as well as at the thermal level. Use the following fast groups:

Group	Lethargy Range
1	0 - 3.0
2	3.0 - 8.0
3	8.0 - 15.5
4	15.5 - 16.5
5	16.5 - 18.966
Thermal	18.966 (500°F)

Use age theory to estimate the fuel loading and reactor size; this will give some approximate numbers to use in computing the group constants. Try to fix things up so that the reactor would run at least ten days at 30 megawatts power, in the age theory approximation.

Set up the finite difference equations one would use in coding this treatment for a computer. Include a criticality computation.

Compute the group constants according to the method given in the class notes (Appendix 1).

TABLE 2.0

	Be			U-235				Water		
Level	Lethargy	Sigma Scatter	Sigma Absorption	Lethargy	Sigma Scatter	Sigma Absorption	σ_f	Lethargy	Sigma Scatter	Sigma Absorption
0	0.000	1.30	0.0000005	0.0	1.30	1.70	1.504	0.000	0.0734064	0.0000011
1	0.500	1.58	0.0000006	0.5	1.90	1.60	1.382	0.5	0.1254582	0.0000014
2	1.000	2.03	0.0000008	1.0	2.50	1.50	1.301	1.0	0.2115439	0.0000018
3	1.500	2.03	0.0000011	1.5	3.00	1.50	1.301	1.5	0.2235558	0.0000023
4	2.000	2.03	0.0000014	2.0	3.50	1.50	1.220	2.00	0.3423407	0.0000030
5	2.500	4.42	0.0000017	2.5	3.70	1.60	1.220	2.5	0.4494971	0.0000039
6	3.000	3.57	0.0000022	3.0	3.80	1.70	1.301	3.0	0.5472112	0.0000050
7	3.500	4.15	0.0000029	3.5	5.10	1.90	1.504	3.5	0.6656625	0.0000064
8	4.000	4.72	0.0000037	4.0	6.40	2.00	1.585	4.0	0.7924553	0.0000082
9	6.000	6.00	0.0000101	6.0	8.50	3.30	2.805	6.0	1.3196466	0.0000222
10	8.000	6.00	0.0000274	8.0	8.50	5.80	4.512	8.0	1.4297562	0.0000604
11	10.000	6.00	0.0000746	10.0	8.50	14.5	10.000	10.0	1.4597861	0.0001643
12	12.000	6.00	0.0002030	12.0	8.50	65.0	38.008	12.0	1.4597861	0.0004472
13	14.000	6.00	0.0005510	14.0	8.50	116.0	67.886	14.0	1.4614544	0.0012138
14	15.500	6.00	0.0010370	15.5	8.50	39.0	23.008	15.5	1.4948210	0.0022844
15	16.500	6.00	0.0019910	16.5	8.50	87.0	76.016	16.5	1.5014943	0.0043859
16	17.500	5.83	0.0030010	17.5	8.50	202.0	163.008	17.5	1.8218131	0.0066108
17	18.500	5.02	0.0052290	18.5	8.50	296.0	253.008	18.5	2.4157374	0.0115187
18	19.795	6.00	0.0092950	19.795	9.00	691.0	583.740	19.795	3.2565742	0.0204756
500	18.966	5.86	0.0065760	18.966	8.50	429.4	359.065	18.966	2.6519726	0.0144878
	← Microscopic in Barns →							← Macroscopic, cm^{-1} →		

APPENDIX 1: MULTIGROUP EQUATIONS

Consider the lethargy range $(0, u_t)$ to be divided into n intervals of width $U_1, \ldots, U_n$ such that $\xi << U_i$ for all i. The average number of collisions a neutron must experience to traverse the i-th group, in the case of pure scattering is

$$A = (U_i/\xi) \frac{\text{Collisions}}{\text{neutron}} \tag{1}$$

Let Σ_i be the "average" cross section for the i-th group and $\Phi_i(x)$ the neutron flux for the group. The number of neutron collisions per cm^3-sec is then

$$c = \Sigma_i \Phi_i(x) \frac{\text{Collisions}}{cm^3\text{-sec}} \tag{2}$$

The ratio c/A is, therefore, the number of neutrons which scatter through the i-th group per unit volume per sec.

$$c/A = \xi \Sigma_i \Phi_i(x)/U_i \tag{3}$$

In the base of pure scattering the diffusion balance equation for the i-th group is

$$D_i \Delta \Phi_i(x) - \frac{\xi \Sigma_i}{U_i} \Phi_i(x) + \frac{\xi \Sigma_{i-1}}{U_{i-1}} \Phi_{i-1}(x) + S_i(x) = 0 \tag{4}$$

for a steady state. The source term has the form

$$S_i(x) = \nu F_i^* \sum_{j=1}^{n} \Sigma_{fi} \Phi_i(x) \tag{5}$$

where

$$F_i^* \equiv \int_{U_i} f(u)\, du$$

f(u) being the p.d.f. for fast neutron lethargy at fission. In the present notation the thermal group is the n-th group which is considered to have zero width, i.e., monolethargic. The thermal balance equation is

$$D_n \Delta \Phi_n(x) - \Sigma_{an} \Phi_n(x) + \frac{\xi \Sigma_{n-1}}{U_{n-1}} \Phi_{n-1}(x) = 0 \tag{6}$$

Note there is no fission source term for the thermal level.

One can account for absorption by altering one term in Equation (4). When absorption is present c/A, in Equation (3), represents the removal of neutrons from the i-th group by either scattering or absorption and hence will still serve as a loss term. The scattering entrance term for neutrons slowing down out of the (i-1)th group is

$$\frac{\xi \Sigma_{s\,i-1}}{U_{i-1}} \Phi_{i-1}(x) \tag{7}$$

and so the diffusion balance equation becomes

$$D_i \Delta \Phi_i(x) - \frac{\xi \Sigma_i \Phi_i(x)}{U_i} + \frac{\xi \Sigma_{s\,i-1} \Phi_{i-1}(x)}{U_{i-1}} + \nu F_i^* \sum_{j=1}^{n} \Sigma_{fj} \Phi_j(x) = 0 \tag{8}$$

The parameters $\{D_i, \Sigma_i, \Sigma_{si}\}$ are averages of $D(u)$, $\Sigma(u)$ and $\Sigma_s(u)$ over the groups $i=1, \ldots, n$. One way to specify the particular way in which the average is performed is to require that:

1. $\langle r^2 \rangle_i$ for the i-th group is that given by the diffusion point kernel
2. The total current for the i-th group is given by

   $D_i \vec{\nabla} \Phi_i(x)$

3. The total number of neutrons passing from the (i-1)th group to the i-th group is consistent with the absorption escape probability for the (i-1)th group.

Condition (1): In diffusion theory one has

$$D = \Sigma_a L^2 = \Sigma_a \langle r^2 \rangle / 6 \tag{9}$$

In the group balance equation the analog to Σ_a is the removal cross section

$$\Sigma_i \equiv \frac{\xi \Sigma_i}{U_i} \tag{10}$$

and so the $\langle r^2 \rangle_i$ condition requires that

$$(6\, U_i\, D_i / \xi\, \Sigma_i) = \langle r^2 \rangle_i \tag{11}$$

It can be shown that (Weinberg and Wigner p. 330)

$$\langle r^2 \rangle_i = \int_{U_i} \frac{6\, D(u)}{\xi\, \Sigma(u)}\, du$$

Hence one arrives at the condition that the quotient D_i/Σ_i has the form

$$\frac{D_i}{\Sigma_i} = \frac{1}{U_i} \int_{U_i} \frac{D(u)}{\Sigma(u)}\, du \tag{12}$$

Condition (2): The current requirement is met by averaging D_i on the basis of flux weighting.

$$\int J(x,u)\, du = -\, D_i \int \vec{\nabla}\, \Phi(x,u)\, du = -\int D(u)\, \vec{\nabla}\, \Psi(x)\, \varphi(u)\, du$$

$$\Rightarrow D_i\, \vec{\nabla}\, \Psi(x) \int \varphi(u)\, du = \vec{\nabla}\, \Psi(x) \int D(u)\, \varphi(u)\, du$$

$$\Rightarrow D_i = \int D(u)\, \varphi(u)\, du \Big/ \int \varphi(u)\, du \tag{13}$$

In the asymtotic form

$$\Phi(E) = F(E)/\Sigma(E) \left(\frac{\text{Const.}}{\xi\, E}\right) \frac{1}{\Sigma(E)}$$

$$\Phi(E)\, dE = \frac{\text{Const.}}{\xi\, E}\, \frac{dE}{\Sigma(E)}$$

$$\Phi(u)\, du = \frac{\text{Const.}}{\xi\, \Sigma(u)}\, du$$

Hence

$$D_i = \int_{U_i} [D(u)/\Sigma(u)]\, du \Big/ \int_{U_i} du/\Sigma(u) \tag{14}$$

But from Equation (12)

$$\int_{U_i} [D(u)/\Sigma(u)]\,du = D_i U_i/\Sigma_i$$

so

$$1 = U_i/\Sigma_i \int_{U_i} du/\Sigma(u) \tag{15}$$

or

$$\frac{1}{\Sigma_i} = \frac{1}{U_i}\int_{U_i}\frac{du}{\Sigma(u)} \tag{16}$$

D_i and Σ_i are thus computed from Equation (14) and Equation (16), respectively.

Condition (3): This condition determines how to compute Σ_{si} so that its effect on the computation will be consistent with the interpretation of Σ_i. This will be the case if

$$\frac{\text{Scattering Loss}}{\text{Scat. Loss + Abs. Loss}} = p_i$$

i.e.

$$\frac{\xi\,\Sigma_{si}\,\Phi_i(x)/U_i}{\xi\,\Sigma_i\,\Phi_i(x)/U_i} = p_i$$

Hence

$$\Sigma_{si} = \Sigma_i\, e^{-\int_{U_i}\frac{\Sigma_a(u)}{\Sigma(u)}\,du} \tag{17}$$

Integration of Multigroup Equations

Having obtained Σ_i, D_i, and Σ_{si} one integrates Equation (8) by assuming a spatial distribution for $\Phi_{ther}(x)$ and then solving Equation (8) for Φ_1 then Φ_2 --- etc down through $\Phi_{ther}(x)$ again. If $\Phi_t^{(0)} = \Phi_t^{(1)}$ the problem is solved; if not one reiterates to get

$$\Phi_1^{(2)} \text{ --- } \Phi_n^{(2)},\ \Phi_t^{(2)}$$

until $\Phi_i^{(m)} = \Phi_i^{(m-1)}$. When this occurs the problem is solved. The criticality constant is the ratio of the neutron populations in successive generations, provided the spatial variation of the flux from one generation to the next remains the same. Once the fluxes remain proportional to each other the criticality factor is the ratio of the fluxes for the same group for two consecutive iterative solutions.

APPENDIX 2: PERTURBATION THEORY

Perturbation theory is used to see what effect small changes in reactor composition have upon its operating characteristics. It is useful in calculating:

1. Effectiveness of thin control rods
2. Mean generation time
3. Criticality change due to fuel depletion
4. Criticality change due to poisons

A detailed discussion of perturbation theory is beyond the scope of these lectures. The general idea, however, is this: A realistic reactor can be thought of as a "perturbation" of an idealistic reactor model for which, at least, one can obtain a solution for flux, criticality, etc. The characteristics of the idealized model are taken as the "unperturbed" state, and the characteristics of the realistic reactor derived using perturbation theory. The perturbations being the differences between the realistic and idealized systems.

a. Importance Function $\Psi^*(\vec{r})$ - Control Rod Placement

The importance function $\Psi^*(\vec{r})$ tells one how many daughter neutrons an originally introduced neutron at position $\vec{r}$ will ultimately supply. Intuitively think of the situation this way: A neutron born at the surface of a reactor has a good chance of escaping. It can't be as important a producer of daughter neutrons, therefore, as a neutron born at the center of the reactor. In one group theory the importance function is proportional to the flux and can, indeed, be thought of as being identical to the flux. We will restrict ourselves to this special case. In this restricted sense the statistical weight W of a volume element $\Delta\vec{r}$ is

$$W(\Delta\vec{r}) = \frac{\Phi^2(\vec{r})\Delta\vec{r}}{\int_{\text{Reactor}} \Phi^2(\vec{r})d\vec{r}}$$

and the criticality change δC arising from a change $\delta\Sigma_a$ in the non-fission absorption cross section in $\Delta\vec{r}$ is

$$\delta C = -\frac{\delta\Sigma_a \Delta\vec{r}\Phi^2(\vec{r})}{\int \nu\Sigma_f \Phi^2(\vec{r})\, d\vec{r}} \propto \delta\Sigma_a W(\Delta\vec{r})$$

In a bare cube reactor, a volume element ΔV at the center of the cube has statistical weight

$$W(\Delta V) = 8\Delta V/V$$

where V is the cube volume. Hence a change $\delta\Sigma_a$ in this central element ΔV is eight times as effective in changing C as the same total change distributed uniformly over the entire volume. This hypersensitivity of a reactor to conditions at its center makes a centrally placed control rod the most effective control rod, provided it is not fully inserted! It turns out that the rate of change in C for a unit length of control rod movement is greatest when the control rod is inserted with its tip at the center of the reactor. Hence regulating rods are usually only inserted half-way to maximize response time. In the above treatment Φ is the unperturbed flux in a homogeneous bare reactor prior to the unposition of the perturbation $\delta\Sigma_a$.

b. Effect of Fuel Depletion

$$\delta C_{\text{depletion}} = -\frac{c\int \Phi(\vec{r})[\nu - 1 - \Sigma_a^{\text{fuel}}/\Sigma_f]\,\Sigma_f[\Phi^2(\vec{r})]\,d\vec{r}}{\int \Phi^2(\vec{r})\,\nu\Sigma_f\,d\vec{r}}$$

(c is a constant)

c. Reciprocal Lifetime

$$\frac{1}{\ell} = \frac{\int[\Phi^2(r)/\ell(\vec{r})]\,d\vec{r}}{\int \Phi^2(r)\,d\vec{r}}$$

$$\frac{1}{\ell(r)} = v\Sigma_a - \frac{v}{\Phi}\,dw\,(D\,\vec{\nabla}\Phi)$$

v = neutron velocity

d. General Change in C: δC

$$\delta C = \frac{\int d\vec{r}\,[\Phi^2(\vec{r})\{\nu\delta\Sigma_f - \delta\Sigma_a\} - \vec{\nabla}\,\Phi(\vec{r})\cdot\delta D\,\vec{\nabla}\Phi(\vec{r})]}{\int \nu\Sigma_f\,\Phi^2(r)\,d\vec{r}}$$

This shows absorption perturbation effects go as Φ^2 but diffusion perturbation effects go as $(\nabla\Phi)^2$.

APPENDIX 3: ELASTIC COLLISIONS
(Neutron Slowing Down Mechanism)

Neutrons can penetrate matter over large distances. Only by direct collisions with nuclei can neutrons be affected in any way. Electromagnetic and electrostatic forces do not affect it. The force of gravity has a negligible affect.

In 1934, Fermi, who was in the habit of storing his neutron source in a fish pond, discovered quite by accident, that the radioactivity produced in targets bombarded with neutrons is greatly increased when the neutrons are passed through a hydrogeneous material. He showed that the neutrons were slowed down in this material, without being absorbed strongly, and that the targets had approximately 1/v type cross sections, increasing the reaction probability.

For materials of low atomic weight, the neutron slowing-down mechanism is elastic scattering. To examine the qualitative and quantitative aspects of scattering by light nuclei, the center of mass (C) system of coordinates is resorted to since the results appear in relatively simple form. Whereas in the laboratory system (L) the target nucleus is assumed to be at rest before the collision, in the C system the center of mass of the neutron and nucleus is assumed to be at rest. The collision process is then described from the viewpoint of an observer moving with the center of mass. Figure A3.1 shows the relationship between the two systems pictorially.

In the L system before collision, the neutron of mass m moves toward the nucleus with speed v_o, momentum mv_o and energy E_o. The nucleus of mass M_t is assumed to be at rest. From the definition of center of mass:

$$R_c = \frac{mr + M_t R}{m + M_t}.$$

Since $\dot{R} = 0$,*

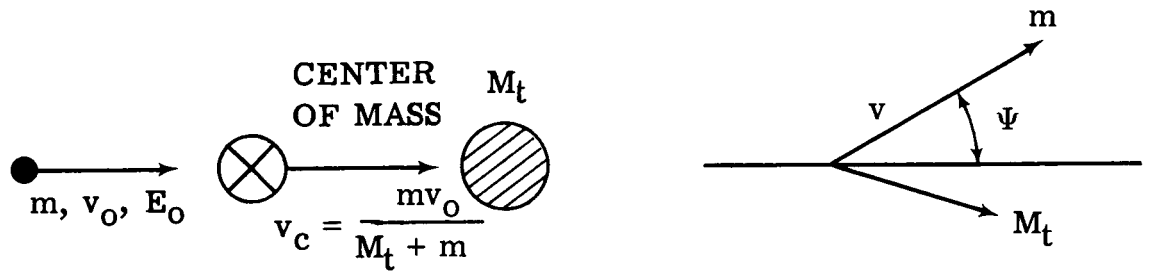
$$\dot{R}_c = \frac{m\dot{r}}{m + M_t} = \frac{mv_o}{m + M_t} = v_c. \qquad \text{(A3.1)}$$

Here v_c is the velocity of the center of mass.

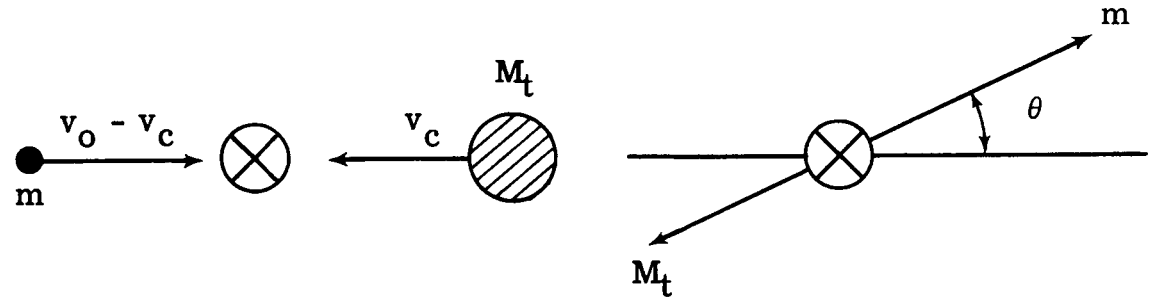
After the collision, the neutron moves with speed v at some angle φ with the original direction.

*$\dot{R} \equiv \frac{d}{dt} R$

Relationship of laboratory system to the center of mass system



(a) L system before collision (b) L system after collision



(c) C system before collision (d) C system after collision

Fig. A3.1 - Neutron slowing down mechanism

In the C system before collision the neutron moves to the right with speed

$$v_o - v_c = v_o \frac{M_t}{M_t + m} \qquad \text{(A3.2)}$$

and the nucleus moves to the left with speed v_c. Thus the total momentum of the C system is

$$m \left(\frac{M_t v_o}{M_t + m}\right) - M_t \left(\frac{m v_o}{M_t + m}\right) = 0. \qquad \text{(A3.3)}$$

A vector diagram of the relationship between the two coordinate systems is shown in Figure A3.2.

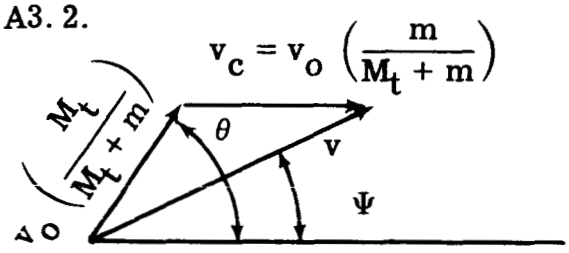


Fig. A3.2 - Vector diagram relating neutron velocity and angle in L and C systems

In the C system, since the total momentum is conserved and is zero, when the neutron is scattered to an angle θ, the nucleus must travel in the direction $180^o + \theta$. For example, let us look at the two extreme cases of elastic scattering:

(a) A glancing collision i.e. $\theta = 0$. By applying the vector addition in Figure A3.2:

$$v = v_o \left(\frac{M_t}{M_t + m}\right) + v_o \left(\frac{m}{M_t + m}\right) = v_o$$

Thus $E = E_o$

(b) A head-on collision i.e., $\theta - 180^o$. By the same method,

$$v = v_o \frac{M_t}{M_t + m} - \frac{m v_o}{M_t + m} = v_o \left(\frac{M_t - m}{M_t + m}\right)$$

$$\text{Thus } \frac{E}{E_o} = \frac{1/2 m v^2}{1/2 m v_o^2} = \left(\frac{M_t - m}{M_t + m}\right)^2$$

In this collision the neutron suffers its maximum energy loss. Note, this energy change is in the L system. In the C system, the neutron energy does not change.

From example (b) we see that for a neutron scattering in graphite (M=12), $[E/E_o]$ max = 0.72. In hydrogen, $[E/E_o]$ max = 0. The neutron then, can lose up to 28 percent of its energy in a collision with a graphite atom and up to 100 percent of its energy in a collision with hydrogen.

Applying the law of cosines to the vector diagram Figure A3.2, we see that:

$$v^2 = v_o^2 \left(\frac{M_t}{M_t+m}\right)^2 + v_o^2 \left(\frac{m}{M_t+m}\right)^2 + 2v_o^2 \left(\frac{M_t}{M_t+m}\right)\left(\frac{m}{M_t+m}\right) \cos\theta \tag{A3.4}$$

The energy ratio is then, in terms of θ:

$$E/E_o = \frac{v^2}{v_o^2} = \frac{M_t^2 + m^2 + 2M_t m\cos\theta}{(M_t+m)^2} \tag{A3.5}$$

Letting $M_t/m = A$,

$$E/E_o = \frac{A^2 + 1 + 2A\cos\theta}{(A+1)^2} \tag{A3.6}$$

The law of sines can now be applied to find the relationship of θ to Ψ:

$$\frac{\sin(\theta - \Psi)}{v_o\left(\frac{m}{M_t+m}\right)} = \frac{\sin\Psi}{v_o\left(\frac{M_t}{M_t+m}\right)} \tag{A3.7}$$

Expanding $\sin(\theta - \Psi)$ we arrive at the result:

$$\tan\Psi = \frac{A\sin\theta}{1 + A\cos\theta} \tag{A3.8}$$

We have now in equations (A3.6) and (A3.8) the neutron energy decrement per collision as a function of A and Ψ.

What we would like to calculate now is - after a collision from an initial energy E_o, what is the probability that a neutron has an energy E? Or, given a large number of neutrons scattering from energy E_o, what is the average energy of the scattered neutrons.

Assuming that in the C system the scattering probability is isotropic, the number of neutrons, dN that will have energy E after a collision is equal to the number scattered to angles between θ and $\theta + d\theta$. This is simply the area on a sphere between θ and $\theta + d\theta$ divided by the total area:

$$dN = N \frac{2\pi r (\sin\theta) r d\theta}{4\pi r^2} = \frac{N}{2} \sin\theta \, d\theta \qquad \text{(A3. 9)}$$

Differentiating equation (6. 23) with respect to θ,

$$dE = - E_o \frac{2A \sin\theta}{(A+1)^2} d\theta \qquad \text{(A3. 10)}$$

The isotropy of scattering in the C system is the essential reason for introducing that system. In the L system, the scattering is predominantly forward, with $(\cos\varphi)_{ave} = 2/3A$.

Substituting N for θ from Equ. (A3. 9) into A3. 10) we find that

$$\frac{dN}{N} = - \frac{(A+1)^2}{4A} \frac{1}{E_o} dE \qquad \text{(A3. 11)}$$

is the probability that a neutron with initial energy E_o will have an energy between E and E + dE. The average logarithmic energy decrement $\langle\xi\rangle$ is a term which we find useful in shielding and reactor problems.

$$\xi \equiv \left\langle \ln \frac{E_o}{E} \right\rangle \qquad \text{(A3. 12)}$$

$$\left\langle \ln \frac{E_o}{E} \right\rangle = \int_{E_{max}}^{E_{min}} \ln \frac{E_o}{E} \frac{dN}{N} = \int_{\frac{(A-1)^2}{(A+1)^2} E_o}^{E_o} \frac{(A+1)^2}{4A} \frac{1}{E_o} \ln \frac{E_o}{E} dE \qquad \text{(A3. 13)}$$

This is easily integrated by substituting $x = E/E_0$ giving the result:

$$\xi = 1 + \frac{(A-1)^2}{2A} \ln \frac{A-1}{A+1} \qquad \text{(A3.14)}$$

Elements having ξ as large as possible have the best slowing down properties. For $A > 10$ the approximation

$$\xi = \frac{2}{A+2/3} \qquad \text{(A3.15)}$$

is accurate to at least 1 percent. Some representative values of ξ are shown in Table A3.1.

TABLE A3.1

Material	A	
H	1	1
D	2	0.725
Be	9	0.209
C	12	0.158
U	238	0.0084
H_2O	18	0.948

To calculate an average $\langle\xi\rangle$ for mixtures, or compounds, assuming molecular binding effects are negligible we can use the formula:

$$\langle\xi\rangle = \frac{\xi_1\Sigma_{s1} + \xi_2\Sigma_{s2} + - - - -}{\Sigma_{s1} + \Sigma_{s2} + - - - -} \qquad \text{(A3.16)}$$

where ξ_i are the logarithmic energy decrements for the pure elements and Σ_{si} are the macroscopic scattering cross sections for the elements in the mixture. In this manner it is calculated that $\xi(H_2O) = 0.948$, $\xi(D_2O) = 0.570$, $\xi(BeO) = 0.173$.

Materials having large ξ and large Σ_s are the more efficient slowing down media. If furthermore, a material has a low Σ_a, it is a good moderator. This leads to the "figure of merit" called the moderating ratio = $\xi\Sigma_s/\Sigma_a$, shown in Table A3.2. Physically the moderating ratio is ξ times the ratio q the scattering probability to the absorption probability, i.e., ξ(prob. to scatter)/(prob. to be absorbed).

TABLE A3. 2

Moderator	Moderator Properties $\xi\Sigma_s$	$\xi\Sigma_s/\Sigma_a$
Water	3.27 cm^{-1}	149
Heavy Water	0.256	7760
Helium (N. T. P.)	0.9×10^{-5}	45
Beryllium	0.181	146
Carbon	0.061	234

APPENDIX 4: SOME STANDARD KERNELS

1. Moment form of the criticality equation

$C = k P_f P_t$

P_f fast neutron nonescape probability

P_t thermal neutron nonescape probability

By the second fundamental theorem of reactor theory

$$P_f = K^*(B) = \int K(r)\, 4\pi r^2 \frac{\sin Br}{Br}\, dr$$

As $B \to 0$ $\frac{\sin Br}{Br} \to 1$ and $P_f = 1$

i.e. $\int_0^\infty K(\underline{x}, \underline{x}')\, dx = 1$ in an <u>infinite</u> medium

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \ldots.$$

Two terms in the series give

$$P_f = K^*(B) \cong \int 4\pi r^2 K(r) \left(1 - \frac{1}{6} B^2 r^2\right) dr$$

$$\cong \int 4\pi r^2 K(r)\, dr - \frac{B^2}{6} \int r^2 K(r)(4\pi r^2\, dr)$$

$$P_f \cong 1 - B^2 \frac{\langle r^2 \rangle}{6} = 1 - B^2 \tau$$

This is valid as long as $\frac{\sin Br}{Br} \cong 1 - Br$ i.e. very small B

All odd moments vanish, and in series form,

$$P_f = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} B^{2n} \langle r^{2n} \rangle$$

2. Gaussian Kernel

$$K(\underline{x}, \underline{x}') = \frac{1}{4\pi\tau}^{3/2} e^{-|\underline{x}-\underline{x}'|^2/4\tau}$$

$$K^*(B) = e^{-B^2\tau} \quad \text{(Age theory)}$$

3. Single Collision Kernel

$$K(\underline{x}, \underline{x}') = \Sigma_o \frac{1}{4\pi|\underline{x}-\underline{x}'|^2} e^{-\Sigma_o|\underline{x}-\underline{x}'|}$$

$$K^*(B) = \frac{\Sigma_o}{B} \tan^{-1}\frac{B}{\Sigma_o} \quad \left| \int K(\underline{x}, \underline{x}')|\underline{x}-\underline{x}'|^2 dx = 6\tau \right.$$

$$\left. = \frac{2}{\Sigma_o^2} \Rightarrow \Sigma_o^2 = \frac{1}{3\tau} \right.$$

$$= \sqrt{\frac{1}{3\tau B^2}} \tan^{-1}\sqrt{3\tau B^2}$$

4. Diffusion Kernel

$$K(\underline{x}, \underline{x}') = \frac{\kappa^2}{4\pi|\underline{x}-\underline{x}'|} e^{-\kappa|\underline{x}-\underline{x}'|}, \quad \kappa^2 = \frac{1}{\tau}$$

$$K^*(B) = \frac{1}{1+\tau B^2}$$

5. Convolution Kernels

In water $K(\underline{x}, \underline{x}')$, Single collision, is best for E > 100 kev and $K(\underline{x}, \underline{x}')$, Gaussian, is best for E < 100 kev.

The slowing down kernel is then

$$K_{H_2O}(\underline{x}, x_o) = \int K_{SC}(x', x_o)\, K_G(x', x')\, d\underline{x}'$$

$$P_f = K^*_{SC}(B)\, K^*_G(B)$$

$$= \sqrt{\frac{1}{3\tau_2 B^2}} \tan^{-1} \sqrt{3\tau_1 B^2} \Big| \; e^{-\tau_2 B^2}$$

τ_1 : 2 Mev to 100 kev

τ_2 : 100 kev to thermal